The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #4

1. Find an equation of the tangent line to

$$\sqrt{x^2 - y} = \frac{9x}{y} - 1$$

at the point P $(5, 9)$.

2. Find a general antiderivative of $f(x) = \dfrac{5\sqrt{x} - 6x^3 - 8x^2 + 3}{x^2}$ for $x > 0$.

3. Solve $y'' = 2\sin(\pi - 2x)$ with $y'(\pi) = 0$ and $y(\pi) = 3$.

4. Find the second derivative of $f(x) = x\cos(x^2)$.Simplify.

5. Find y' in terms of x and y if $\quad 2x + 3y = \dfrac{y^2}{x+1}$.

6. Find a general antiderivative of $\quad f(x) = \dfrac{1}{\cos^2(3x - 1)}$ in the domain (find the domain).

7. Solve $y'' = \dfrac{3}{\sqrt{x}} - 6x$, $y'(4) = 2, y(4) = 0$.

8. Find the second derivative of $y = \dfrac{1}{1 + x^2}$. Simplify.

9. Find an equation of the tangent line at the point $(6, \pi)$ to

$$2\cos\frac{y}{x} + \frac{xy}{6} = \sqrt{3} + \pi.$$

10. Solve (i.e. find y including an interval)

$$y' = \frac{1}{(5 - x)^3}$$

with $\quad y(4) = 1$

11. Find $\quad \displaystyle\int \left(3\sqrt{x} - \frac{1}{3x}\right)^2 dx$ for $x > 0$.

SOLUTIONS
For 1)
Use implicit differentiation and Chain Rule on the left ,Quotient Rule on right:
$\frac{1}{2}\left(x^2 - y\right)^{-\frac{1}{2}} \cdot \left(x^2 - y\right)' = 9 \cdot \left(\dfrac{x}{y}\right)'$

$\frac{1}{2}\left(x^2-y\right)^{-\frac{1}{2}}\cdot\left(2x-y'\right)=9\cdot\dfrac{1\cdot y-x\cdot y'}{y^2}$
now,$x=5,y=9,y'=m$
$\frac{1}{2}\left(25-9\right)^{-\frac{1}{2}}\left(10-m\right)=9\cdot\dfrac{9-5m}{9^2}$ so $\frac{1}{8}\left(10-m\right)=\frac{1}{9}\left(9-5m\right)$
multiply by $9\cdot 8$
$90-9m=72-40m$ thus $31m=-18$ and $m=-\frac{18}{31}$and an equation is
$y=-\frac{18}{31}\left(x-5\right)+9$.
For 2)
$\int f(x)dx=5\int\frac{\sqrt{x}}{x^2}dx-6\int\dfrac{x^3}{x^2}dx-8\int\dfrac{x^2}{x^2}dx+3\int x^{-2}dx=$
$5\int x^{-\frac{3}{2}}dx-6\int xdx-8\int dx+3\cdot\frac{x^{-1}}{-1}+c=5\cdot(-2)\,x^{-\frac{1}{2}}-6\cdot\frac{x^2}{2}-8x-\dfrac{3}{x}+c$
$=-\dfrac{10}{\sqrt{x}}-3x^2-8x-\dfrac{3}{x}+c$ for $x>0$.
For 3)
$y'=\int y''dx=2\int\sin\left(\pi-2x\right)dx=2\cdot\frac{-\cos(\pi-2x)}{-2}+c_1=\cos\left(\pi-2x\right)+c_1$
using $\int\sin(ax+b)dx=\dfrac{-\cos(ax+b)}{a}+c$
now use the condition $y'=0$ for $x=\pi$
$0=\cos\left(-\pi\right)+c_1=-1+c_1$ so $c_1=1$ and $y'=\cos\left(\pi-2x\right)+1$
using $\int\cos(ax+b)dx=\dfrac{\sin(ax+b)}{a}+c$
$y=\int y'dx=\int\cos\left(\pi-2x\right)dx+\int 1dx+c_2=\frac{\sin(\pi-2x)}{-2}+x+c_2=-\frac{1}{2}\sin\left(\pi-2x\right)+x+c_2$
use the second condition $y=3$ for $x=\pi$
$3=-\frac{1}{2}\sin\left(-\pi\right)+\pi+c_2=\pi+c_2$ so $c_2=3-\pi$ and
the solution is $y=-\frac{1}{2}\sin\left(\pi-2x\right)+x+3-\pi$
For 4)
by Product and Chain Rules
$f'(x)=\left[x\cos(x^2)\right]'=1\cdot\cos\left(x^2\right)+x\left(-\sin x^2\right)2x=\cos\left(x^2\right)-2x^2\sin(x^2)$
again
$f''(x)=\left[\cos\left(x^2\right)-2x^2\sin(x^2)\right]'=-2x\sin(x^2)-4x\sin(x^2)-2x^2\cos(x^2)2x=$
$=-6x\sin(x^2)-4x^3\cos(x^2)$
For 5)
use implicit differentiation,Quotient and Chain Rules:
$2+3y'=\dfrac{2yy'\left(x+1\right)-y^2}{(x+1)^2}$ multiply both side by $\left(x+1\right)^2$
$2\left(x+1\right)^2+3y'\left(x+1\right)^2=2yy'(x+1)-y^2$ all terms with y'
$y'\left[3\left(x+1\right)^2-2y\left(x+1\right)\right]=-y^2-2(x+1)^2$
so

$$y'=\frac{-y^2-2(x+1)^2}{3\left(x+1\right)^2-2y\left(x+1\right)}\text{ if the denominator is not 0.}$$

OR
we can simplify first by multiplying by $(x+1)$ to get
$2x^2+2x+3xy+3y=y^2$ then differentiate by Pr.rule
$4x+2+3y+3xy'+3y'=2yy'\qquad 4x+2+3y=y'(2y-3x-3)$
then

$$y' = \frac{4x+2+3y}{2y-3x-3} \text{ if the denominator is not 0.}$$

Note, that it looks different because we have a relation between x and y .

For 6)

$\int \frac{1}{\cos^2(3x-1)}dx = \frac{1}{3}\tan(3x-1)+c \qquad \text{since } (\tan x)' = \sec^2 x = \frac{1}{\cos^2 x}$

if $3x-1 \neq \frac{\pi}{2}+k\pi$ so $x \neq \frac{1}{3}+\frac{\pi}{6}+k\frac{\pi}{3}, k = 0, \pm1, \pm2, \pm3, ...$

For 7)

$y'' = \frac{3}{\sqrt{x}} - 6x \qquad y'(4) = 2 \qquad y(4) = 0.$

$y' = \int y'' dx = 3\int x^{-\frac{1}{2}}dx - 6\int x dx + c_1 = 6\sqrt{x} - 3x^2 + c_1$

now $x = 4 \qquad y' = 2 \qquad$ solve for c_1 :

$2 = 6 \cdot 2 - 3 \cdot 16 + c_1 \qquad c_1 = 38$

so

$$y' = 6\sqrt{x} - 3x^2 + 38 \qquad \text{for } x > 0$$

again

$y = \int y' dx = 6\int x^{\frac{1}{2}}dx - 3\int x^2 dx + 38\int dx = 6 \cdot \frac{2}{3}x^{\frac{3}{2}} - 3 \cdot \frac{x^3}{3} + 38x + c_2$

$y = 4x^{\frac{3}{2}} - x^3 + 38x + c_2$

now $x = 4 \qquad y = 0 \qquad$ solve for c_2 :

$0 = 4 \cdot 2^3 - 4^3 + 38 \cdot 4 + c_2 \qquad c_2 = -4(8 - 16 + 38) = -120$

thus the solution of the given problem is

$$y = 4x^{\frac{3}{2}} - x^3 + 38x - 120 \text{ for any } x > 0.$$

For 8)

by Chain Rule

$y' = \left(\frac{1}{1+x^2}\right)' = \left[(1+x^2)^{-1}\right]' = (-1)(1+x^2)^{-2}2x = -2x(1+x^2)^{-2}$

by product and chain rules

$y'' = [-2x(1+x^2)^{-2}]' = -2(1+x^2)^{-2} - 2x(-2)(1+x^2)^{-3}2x =$

$= -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$

Or by Quotient Rule

$y' = \left(\frac{1}{1+x^2}\right)' = \frac{0-2x}{(1+x^2)^2} \qquad y'' = \left(\frac{-2x}{(1+x^2)^2}\right)' =$

$\frac{-2(1+x^2)^2 + 2x2(1+x^2)2x}{(1+x^2)^4} = \frac{-2(1+x^2)+8x^2}{(1+x^2)^3} = \frac{-2+6x^2}{(1+x^2)^3}$

For 9)

Use implicit differentiation:

$2\left(-\sin\frac{y}{x}\right)\left(\frac{y}{x}\right)' + \frac{1}{6}(x \cdot y)' = 0$

$-2\sin\frac{y}{x} \cdot \frac{y' \cdot x - y \cdot 1}{x^2} + \frac{1}{6}(1 \cdot y + x \cdot y') = 0$

Now $x = 6, y = \pi$,and $y' = m$:

$-2\sin\frac{\pi}{6} \cdot \frac{6m-\pi}{36} + \frac{1}{6}(\pi + 6m) = 0$, multiply both sides by 36 and use $\sin\frac{\pi}{6} = \frac{1}{2}$

thus

$-(6m-\pi) + 6(\pi+6m) = 0$ and the equation is now:$-6m + \pi + 6\pi + 36m = 0$,

thus

$30m = -7\pi$ and $m = -\frac{7\pi}{30}$.The equation of the tangent line is :

$$y = -\frac{7\pi}{30}(x-6) + \pi$$

.

For 10)

For $x \neq 5$ $\quad y = \int y' dx = \int (5-x)^{-3} dx = \dfrac{(5-x)^{-2}}{-2(-1)} + c = \dfrac{1}{2(5-x)^2} + c$

using $\int (ax+b)^r dx = \dfrac{(ax+b)^{r+1}}{a(r+1)} + c$,where $a = -1, b = 5, r = -3$

now if $x = 4, y = 1$ solve for c: $\quad 1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$.

Together the solution is $y = \frac{1}{2}(5-x)^{-2} + \frac{1}{2}$ for $x \in (-\infty, 5)$

since the condition is at $x = 4 < 5$.

For 11)

$\displaystyle\int \left(3\sqrt{x} - \frac{1}{3x}\right)^2 dx$(get rid of the power using $(A-B)^2 = A^2 - 2AB + B^2$)

$= \displaystyle\int \left[(3\sqrt{x})^2 - 2 \cdot 3\sqrt{x} \cdot \frac{1}{3x} + \left(\frac{1}{3x}\right)^2\right] dx =$

$= 9\int x dx - 2\int x^{-\frac{1}{2}} dx + \frac{1}{9}\int x^{-2} dx = 9 \cdot \frac{1}{2}x^2 - 2 \cdot 2x^{\frac{1}{2}} + \frac{1}{9} \cdot \frac{x^{-1}}{-1} + c = \dfrac{9}{2}x^2 - 4\sqrt{x} - \dfrac{1}{9x} + c$

for $x > 0$.