The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #5

1. How much money one has to invest today at the interest of 3% compounded quaterly to get \$10,000 in 10 years?

2. Find the domain and derivative of $f(x) = (3x)^{\pi} + \pi^{3x} + (\pi x)^{3x}$.

3. Find y' if $y = 2^{x^4} + \frac{2}{x^4} + \left(\frac{1}{x}\right)^x$,for $x > 0$.

4. How long does it take to double your investment if the interest of 7 % is compounded

   (a) (a) yearly? (b) monthly?

5. In the first few weeks after birth,a baby gains weight at a rate proportional to its weight.A baby weighing 4kg at birth weighs 4.4kg after 2 weeks. How much did it weigh 4 days after birth ?

6. For $f(x) = 3^x \ln \frac{3}{x}$ find $f'(3)$.

7. Find y' if $y = x^{x^2} + \ln \frac{1}{1-x}$,for $0 < x < 1$.

8. After 3 days a sample of radon-222 decayed to 58% of its original amount.What is half-life of radon-222?

9. Evaluate $(a)\ \lim_{x\to+\infty} \frac{x}{2^x - 1}$ $(b)\ \lim_{x\to-\infty} \frac{x}{2^x - 1}$ $(c)\ \lim_{x\to 0} \frac{x}{2^x - 1}$

10. Evaluate

    $(a)\ \lim_{x\to\infty} \frac{x^2}{e^{3x}}$ $(b)\ \lim_{x\to-\infty} \frac{x^2}{e^{3x}}$ $(c)\ \lim_{x\to\infty} \frac{(\ln x)^2}{x}$ $(d)\ \lim_{x\to 0^+} \frac{(\ln x)^2}{x}$.

11. Solve for x : $\frac{1}{2^{x+1}} = \frac{5}{4^x}$.

    .

12. Solve for x: $(a)\ \frac{1}{2}\ln(x+3) + 1 = 0;$ $(b)\ 3^{x^2} = 9^{x-3}$.

SOLUTION
For1)

the formula to use is $A(t) = A_0\left(1 + \frac{p}{100n}\right)^{nt}$ where $t = 10, n = 4, p = 3$

$A = 10,000$ $A_0 = ?$ is the initial amount to invest so

$10000 = A_0(1 + \frac{3}{400})^{40} = A_0(\frac{403}{400})^{40}$ multiply by reciprocal to isolate A_0

$A_0 = 10000(\dfrac{400}{403})^{40} = \$7416.48$

For 2) $\quad x > 0$

$f(x) = (3x)^\pi + \pi^{3x} + (\pi x)^{3x} = 3^\pi x^\pi + (\pi^3)^x + e^{3x \ln \pi x}$

power + exp.f + chain rule for e^u so by Chain Rule $(e^u)' = e^u u'$

$f'(x) = 3^\pi \left(x^\pi\right)' + \left[(\pi^3)^x\right]' + e^{3x\ln \pi x}\left[3x\ln(\pi x)\right]'$ (Pr.R.)

$= \pi 3^\pi x^{\pi-1} + (\pi^3)^x \ln \pi^3 + e^{3x\ln \pi x}\left[3\ln \pi x + 3x \cdot \dfrac{1}{\pi x}\cdot \pi\right] =$

$= \pi 3^\pi x^{\pi-1} + 3\left(\pi^3\right)^x \ln \pi + e^{3x\ln\pi x}\left[3\ln \pi x + 3\right]$

ALSO change all terms into e^u and then Chain rule

$f(x) = (3x)^\pi + \pi^{3x} + (\pi x)^{3x} = e^{\pi \ln 3x} + e^{3x\ln \pi} + e^{3x\ln \pi x}$

so

$f'(x) = e^{\pi\ln 3x}\left(\pi \ln 3x\right)' + e^{3x\ln\pi}\left(3x\ln\pi\right)' + e^{3x\ln\pi x}\left(3x\ln \pi x\right)' =$

$= e^{\pi \ln 3x}\left(\pi\dfrac{3}{3x}\right) + e^{3x\ln\pi}\left(3\ln\pi\right) + e^{3x\ln\pi x}\left[3\ln\pi x + 3x\cdot\dfrac{1}{\pi x}\cdot \pi\right] =$

$= \dfrac{\pi}{x}(3x)^\pi + \left(3\ln\pi\right)\pi^{3x} + \left[3\ln \pi x + 3\right](\pi x)^{3x}$

For 3)

$y' = (e^{x^4\ln 2})' + (2x^{-4})' + \left(e^{x\ln\frac{1}{x}}\right)' = e^{x^4\ln 2}\left(x^4\ln 2\right)' - 8x^{-5} + e^{-x\ln x}\cdot\left(-x\ln x\right)' =$

$= 4x^3\ln 2\, e^{x^4\ln 2} - 8x^{-5} - e^{-x\ln x}\left(1\cdot\ln x + x\cdot\frac{1}{x}\right) = 4x^3\ln 2\, e^{x^4\ln 2} - 8x^{-5} - e^{-x\ln x}\left(\ln x + 1\right)$

Or by log.diff.BUT only for the last term $u = \left(\frac{1}{x}\right)^x = e^{x\ln\frac{1}{x}}$ so

$\ln u = x\ln\frac{1}{x} = x\ln x^{-1} = -x\ln x$

and $\frac{1}{u}\cdot u' = -\left(1\cdot\ln x + x\cdot\frac{1}{x}\right) = -(\ln x + 1)$ so $u' = -\left(\frac{1}{x}\right)^x(\ln x + 1)$.

for the first term we can use $(2^u)' = 2^u\cdot\ln 2\cdot u'$ so $\left(2^{x^4}\right)' = 2^{x^4}\cdot\ln 2\cdot 4x^3$

and the second term is just a power so $(2x^{-4})' = -8x^{-5}$.Together

$y' = 4x^3\ln 2\cdot 2^{x^4} - 8x^{-5} - \left(\frac{1}{x}\right)^x(\ln x + 1)$ as above.

For 4a)

we have to solve for t $\quad 2A_0 = A_0\left(1 + \frac{7}{100}\right)^t$ cancel A_0

thus $\quad 2 = (1.07)^t \quad$ take ln of both sides:

$\ln 2 = t\ln 1.07$, so $t = \frac{\ln 2}{\ln 1.07} = 10.24$

We need 10 years and almost 3 months.

For 4b) we have to solve for t $\quad 2A_0 = A_0\left(1 + \frac{7}{1200}\right)^{12t}$

as above $\quad$ thus $2 = \left(\frac{1207}{1200}\right)^{12t}$ and

$\ln 2 = 12t\ln\dfrac{1207}{1200}$,and $t = \frac{\ln 2}{12(0.0058163)} = 9.929$.

So this time we need less than 10 years.

For 5)

for the weight after t weeks $W(t) = 4e^{kt}$, given :$W(2) = 4.4 = 4e^{2k}$

solve for k : $\quad \dfrac{4.4}{4} = e^{2k}$, apply ln to both sides: $\ln 1.1 = 2k$, so $k = \frac{\ln 1.1}{2}$

Now,after 4 days means $\frac{4}{7}$ of a week ,so $W = 4e^{k\frac{4}{7}} = 4e^{\frac{2}{7}\cdot\ln 1.1} = 4.11$ kg.

OR $\quad W(t) = 4e^{kt}$where we measure time t in days then 2 weeks is 14 days

$W(14) = 4.4 = 4e^{14k}$ then $k = \frac{\ln 1.1}{14}$

and for $t = 4$ $\quad W = 4e^{k4} = 4e^{4\frac{\ln 1.1}{14}} =$ (as above)$= 4.11$ kg

For 6)

first simplify $f(x) = 3^x \cdot (\ln 3 - \ln x)$, then use Product Rule

$f'(x) = (3^x)' (\ln 3 - \ln x) + 3^x (\ln 3 - \ln x)' = 3^x \ln 3 (\ln 3 - \ln x) + 3^x \left(\frac{-1}{x}\right)$

since $(\ln 3)' = 0$ then $x = 3$

and $f'(3) = 3^3 \ln 3 (\ln 3 - \ln 3) + 3^3 \left(\dfrac{-1}{3}\right) = -9.$

For 7)

$y = x^{x^2} + \ln \frac{1}{1-x} = e^{x^2 \ln x} - \ln(1-x)$ $\quad y' = e^{x^2 \ln x} \left(x^2 \ln x\right)' - \dfrac{1}{1-x} (1-x)' =$

$= e^{x^2 \ln x} (2x \ln x + x) + \dfrac{1}{1-x}$

OR you can use log.diff. but only for the first part

$u = x^{x^2}$ $\quad \ln u = x^2 \ln x$ $\quad \dfrac{u'}{u} = 2x \ln x + x^2 \cdot \dfrac{1}{x}$ $\quad u' = x^{x^2} (2x \ln x + x)$

and $\quad y' = u' + \dfrac{1}{1-x}.$

For 8)

the correct formula $\quad A(t) = A_0 e^{kt}$ where $k < 0, t$ in days, $A_0 = 100\%$

first info $\quad$ if $t = 3$ $\quad 58 = 100e^{3k}$ solve for k $\quad \ln \dfrac{58}{100} = 3k$

$k = \dfrac{\ln 0.58}{3} = -0.1815757$

so $\quad A(t) = 100e^{kt}$ for k calculated above ;now half-life T means we got 50%

$50 = 100e^{kT}$ $\quad$ where $k = \dfrac{\ln 0.58}{3}$ $\quad \dfrac{50}{100} = e^{kT}$

solve for T $\quad \ln 0.5 = kT$ $\quad T = \dfrac{3 \ln 0.5}{\ln 0.58} = 3.8174$ days

For 9)

for a) $\quad \displaystyle\lim_{x \to +\infty} \frac{x}{2^x - 1}$ $\quad$ the type is "$\frac{+\infty}{+\infty}$" so we can use L'Hop.rule

$\displaystyle\lim_{x \to +\infty} \frac{(x)'}{(2^x - 1)'} = \lim_{x \to +\infty} \frac{1}{2^x \ln 2} =$ "$\frac{1}{\infty}$" $= 0$

for b) $\quad \displaystyle\lim_{x \to -\infty} \frac{x}{2^x - 1}$ since "$2^{-\infty}$" $= 0$ No L'H.R.

$\displaystyle\lim_{x \to -\infty} \frac{x}{2^x - 1} = \frac{-\infty}{-1} = +\infty.$

for c) $\quad \displaystyle\lim_{x \to 0} \frac{x}{2^x - 1}$ the type is "$\frac{0}{0}$" so L'Hop.Rule again

$\displaystyle\lim_{x \to 0} \frac{x}{2^x - 1} = \lim_{x \to 0} \frac{1}{2^x \ln 2} = \frac{1}{\ln 2}$

For 10)

for a) $\quad$ the type is "$\frac{\infty}{\infty}$" so use L'Hop.Rule

$\displaystyle\lim_{x \to \infty} \frac{x^2}{e^{3x}} = \lim_{x \to \infty} \frac{2x}{3e^{3x}}$ (again) $\displaystyle= \lim_{x \to \infty} \frac{2}{9e^{3x}} =$ "$\frac{2}{\infty}$" $= 0$

for b) $\quad$ the type is "$\frac{\infty}{0^+}$" since "$e^{-\infty}$" $= 0$

so No L'H.R. but $\displaystyle\lim_{x \to -\infty} \frac{x^2}{e^{3x}} = \lim_{x \to -\infty} x^2 e^{-3x} = +\infty$

Or "$\frac{1}{0^+}$" $= +\infty.$

for c) $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \text{"}\frac{\infty}{\infty}\text{"(L'H.R)} = \lim_{x\to\infty} \frac{2(\ln x)\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x} =$

again $= \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \text{"}\frac{2}{\infty}\text{"} = 0$

for d) $\lim_{x\to 0^+} \frac{(\ln x)^2}{x} = \text{"}\frac{\infty}{0^+}\text{"} = \lim_{x\to 0^+} (\ln x)^2 \cdot \frac{1}{x} = (-\infty)^2 (+\infty) = +\infty$
(No L'H.R.)

For 11)
cross multiply first,so $4^x = 5 \cdot 2^{x+1}$, then apply ln to both sides
$\ln 4^x = \ln (5 \cdot 2^{x+1}) = \ln 5 + \ln 2^{x+1}$
thus $x \ln 4 = \ln 5 + (x+1)\ln 2 \qquad x\ln 4 = \ln 5 + x\ln 2 + \ln 2$
and $x\ln 4 - x\ln 2 = \ln 5 + \ln 2$
So $x(\ln 4 - \ln 2) = \ln (5\cdot 2)$ and finally $x \ln \frac{4}{2} = \ln 10, x = \frac{\ln 10}{\ln 2}$.

For 12a)
$\frac{1}{2}\ln(x+3) = -1 \qquad \ln(x+3) = -2$
then exp.f. to both sides and $(x+3) = e^{-2}$
and so $x = e^{-2} - 3$.

b)
Take log of both sides: $\ln 3^{x^2} = \ln 9^{x-3}$.
$x^2 \ln 3 = (x-3)\ln 9 = (x-3)\ln 3^2 = (x-3)\cdot 2\ln 3$
cancel $\ln 3$ and $x^2 = 2(x-3) = 2x - 6$
everything on one side :$x^2 - 2x + 6 = 0$, discriminant of this polynomial is
$D = (-2)^2 - 4\cdot 1\cdot 6 = -20$ so no real roots exist and the problem has NO solution.
Also we can change both sides to the same base: $3^{x^2} = (3^2)^{x-3} = 3^{2x-6}$ and by comparing the exponents we get the same quadratic equation.