MATH 251/249
Midterm Handout

1. **For 1)**

$\lim_{x\to\infty}\left(x^2-x^2\cos\frac{1}{x}\right)=\lim_{x\to\infty}x^2\left(1-\cos\frac{1}{x}\right)$ "$\infty\cdot 0$" $\quad u=\frac{1}{x}, u\to 0^+$

so $\lim_{x\to\infty}\left(x^2-x^2\cos\frac{1}{x}\right)=\lim_{u\to 0^+}\frac{(1-\cos u)}{u^2}\cdot\frac{(1+\cos u)}{(1+\cos u)}=\lim_{u\to 0^+}\frac{(1-\cos^2 u)}{u^2(1+\cos u)}=$

$$=\lim_{u\to 0^+}\frac{\sin^2 u}{u^2(1+\cos u)}=\lim_{u\to 0^+}\left(\frac{\sin u}{u}\right)^2\cdot\lim_{u\to 0^+}\frac{1}{(1+\cos u)}=\frac{1}{2}.$$

For 2)

$\lim_{x\to 0}\frac{\sin x}{x-\pi}=\frac{0}{-\pi}=0$

$\lim_{x\to\pi}\frac{\sin x}{x-\pi}=$ "$\frac{0}{0}$" $=\lim_{u\to 0}\frac{\sin(u+\pi)}{u}=\lim_{u\to 0}\frac{\sin u\cos\pi+\cos u\sin\pi}{u}=-1$

where $x-\pi=u$ and $\cos\pi=-1$,$\sin\pi=0$

$\lim_{x\to-\infty}\frac{\sin x}{x-\pi}=0$ by Squeeze Th. since $-1\le\sin x\le 1$ and $\quad x-\pi<0$

$\frac{-1}{x-\pi}\ge\frac{\sin x}{x-\pi}\ge\frac{1}{x-\pi}$ and $\lim_{x\to-\infty}\frac{\pm 1}{x-\pi}=0.$

For 3)

for $y=\frac{\cos\pi x}{1-x}$ use Q.R. to find $y'=\frac{-\pi\sin\pi x(1-x)-\cos\pi x(-1)}{(1-x)^2}$

at $x=-\frac{1}{2}$ $\quad\cos\frac{-\pi}{2}=0,\sin\frac{-\pi}{2}=-1$ so $y=0$ and

$y'=\frac{3\frac{\pi}{2}}{\frac{9}{4}}=\frac{2\pi}{3}=m$... slope $\quad$ tangent $y=\frac{2\pi}{3}\left(x+\frac{1}{2}\right)$

For 4)

for $y=\left(\sin\frac{1}{\sqrt{x^4+1}}\right)^3$ use Chain Rule twice

$y'=3\left(\sin\frac{1}{\sqrt{x^4+1}}\right)^2\left(\cos\frac{1}{\sqrt{x^4+1}}\right)\left((x^4+1)^{-\frac{1}{2}}\right)'=$

$=3\left(\sin\frac{1}{\sqrt{x^4+1}}\right)^2\left(\cos\frac{1}{\sqrt{x^4+1}}\right)\left(-\frac{1}{2}\right)\left((x^4+1)^{-\frac{3}{2}}\right)4x^3=\frac{-6x^3\sin^2\frac{1}{\sqrt{x^4+1}}\cos\frac{1}{\sqrt{x^4+1}}}{(x^4+1)^{\frac{3}{2}}}$

For 5)

the function $f(x)=x-2\sin(\pi x)$ is continuous everywhere

and $f(\frac{1}{2})=\frac{1}{2}-2\sin\frac{\pi}{2}=-\frac{3}{2}<0$ and $f(1)=1-2\sin\pi=1>0$

so by IVT there must be an c between $\frac{1}{2}$ and 1 where $f(c)=0$

For 6)

the polynoial $p=2x^3-6x^2+7$ is continuous everywhere and $\quad p(0)=7>0$

and $p(-1)=-2-6+7=-1<0$ so by IVT(intermediate value theorem)

there must be one root r_1 between -1 and $0 \quad r_1 \in]-1,0[$

since $p(1) = 2 - 6 + 7 = 3 > 0$ and $p(2) = 16 - 24 + 7 = -1 < 0$

so by IVT there is another root between 1 and 2 $\quad r_2 \in]1,2[$

finally $p(3) = 54 - 54 + 7 = 7 > 0$

by IVT there is a root $\quad r_3 \in (2,3)$

For 7)

$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-\frac{1}{25}} = \pm\sqrt{\frac{24}{25}} = \pm\frac{\sqrt{24}}{5}$

but since θ is in the second quadrant cos must be negative

$\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{\sqrt{24}}$

For 8)

$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\frac{4}{9}} = \pm\sqrt{\frac{5}{9}} = \pm\frac{\sqrt{5}}{3}$

but since θ is in the forth quadrant sin must be negative

$\sin\theta = -\dfrac{\sqrt{5}}{3}$. Now $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(\frac{-\sqrt{5}}{3}\right)\cdot\frac{2}{3} = -\dfrac{4\sqrt{5}}{9}$.

For 9)

$$f(x) = \begin{cases} \left(\frac{2}{2x+1} - 3\right)(4x^2-1) & \text{for} \quad x < -\frac{1}{2} \\ ax + b & \text{for} \quad -\frac{1}{2} \le x \le 2 \\ \cos(-\frac{\pi}{x}) & \text{for} \quad x > 2 \end{cases} .$$

the function is continuous except $x = -\frac{1}{2}$ and $x = 2$

$f(-\frac{1}{2}) = \frac{-1}{2}a + b = \lim\limits_{x\to-\frac{1}{2}^+} f(x)$ and $\lim\limits_{x\to-\frac{1}{2}^-} f(x) = \lim\limits_{x\to-\frac{1}{2}^-}\left(\frac{2}{2x+1} - 3\right)(4x^2-1) =$

$\lim\limits_{x\to-\frac{1}{2}^-}\left(\frac{2-3(2x+1)}{2x+1}\right)(4x^2-1)$

$= \lim\limits_{x\to-\frac{1}{2}^-}\left(\frac{2-6x-3}{2x+1}\right)(2x-1)(2x+1) = \lim\limits_{x\to-\frac{1}{2}^-}(-1-6x)(2x-1) = 2\cdot(-2) = -4$

All 3 numbers must be the same so $\quad \frac{-1}{2}a + b = -4$

Similarly,for $x = 2$

$f(2) = 2a + b = \lim\limits_{x\to2^-} f(x)$ and $\lim\limits_{x\to2^+}\cos\frac{-\pi}{x} = \cos\frac{-\pi}{2} = 0$

All 3 numbers must be the same so $2a + b = 0 \longrightarrow b = -2a$

back to $\frac{-1}{2}a + b = -4$ and.$a\left(\frac{-1}{2} - 2\right) = -4 \Rightarrow a\left(\frac{-5}{2}\right) = -4$

so $a = \frac{8}{5}$ and $b = -2a = \frac{-16}{5}$.

For 10)

$$f(x) = \begin{cases} \cos(\pi x) - 2\sin\frac{\pi x}{2} & \text{for} \quad x > 3 \\ ax^2 + b & \text{for} \quad 0 \le x \le 3 \\ 6\cdot\dfrac{\sqrt{9-x}-3}{x} & \text{for} \quad x < 0 \end{cases}$$

the function is continuous except $x = 3$ and $x = 0$

$f(3) = 9a + b = \lim\limits_{x \to 3^-} f(x)$ and

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} \left[\cos(\pi x) - 2\sin\frac{\pi x}{2}\right] = \cos(3\pi) - 2\sin\frac{3\pi}{2} = -1 - 2(-1) = 1$

All 3 numbers must be the same so $9a + b = 1$

Similarly,for $x = 0$

$f(0) = b = \lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^-} 6 \cdot \dfrac{\sqrt{9-x} - 3}{x} \cdot \dfrac{\sqrt{9-x} + 3}{\sqrt{9-x} + 3} = 6 \lim\limits_{x \to 0^-} \dfrac{9 - x - 3^2}{x\left(\sqrt{9-x} + 3\right)} =$

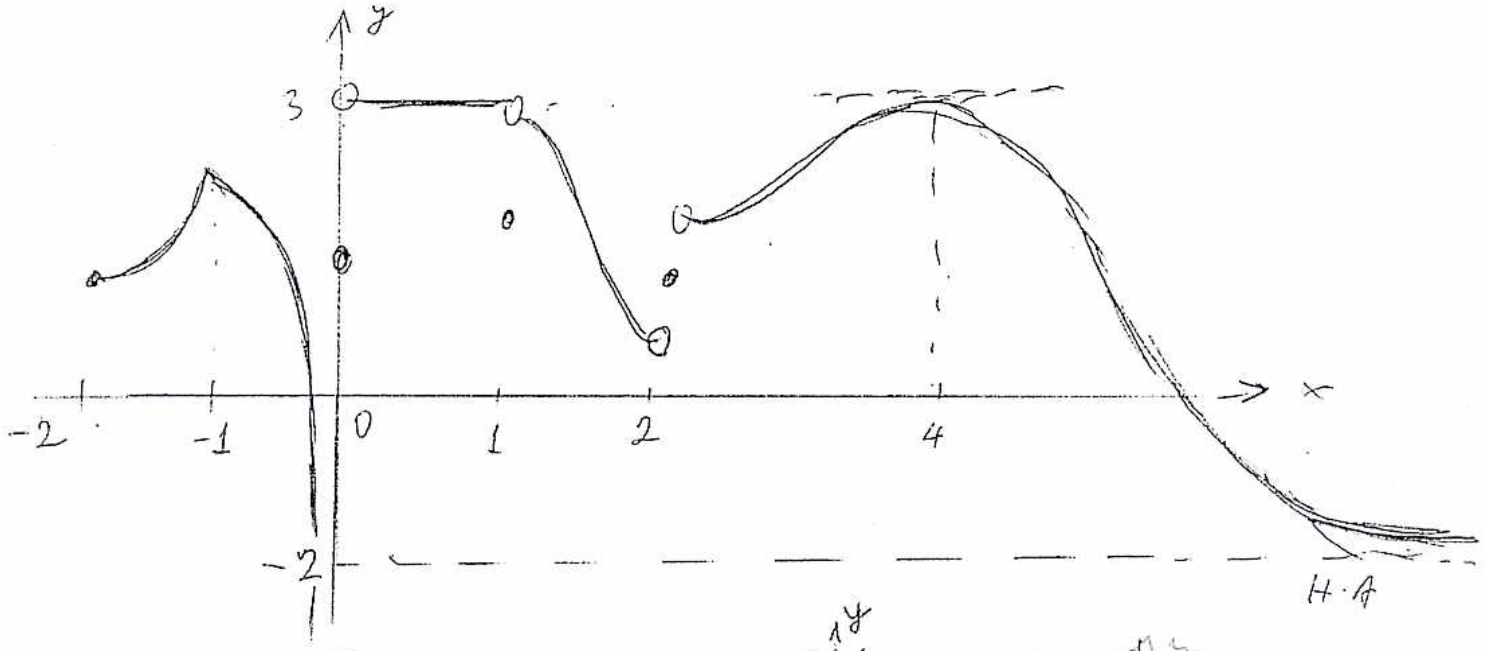
$= 6 \lim\limits_{x \to 0^-} \dfrac{-x}{x\left(\sqrt{9-x} + 3\right)} = 6 \lim\limits_{x \to 0^-} \dfrac{-1}{(\sqrt{9-x} + 3)} = \frac{-6}{6} = -1$

All 3 numbers must be the same so $b = -1$

substitute $b = -1$ in the first equation $\quad 9a - 1 = 1 \quad a = \frac{2}{9}$ and $b = -1$.

For 11)

A



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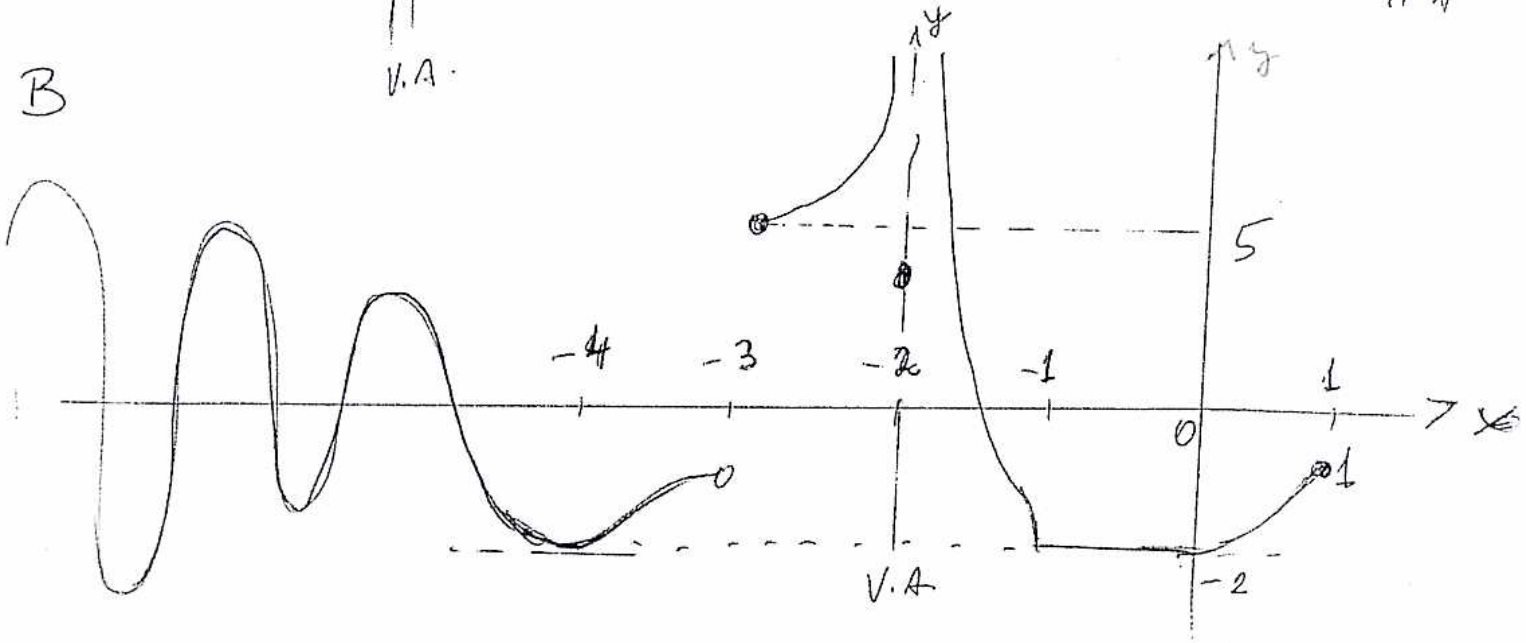

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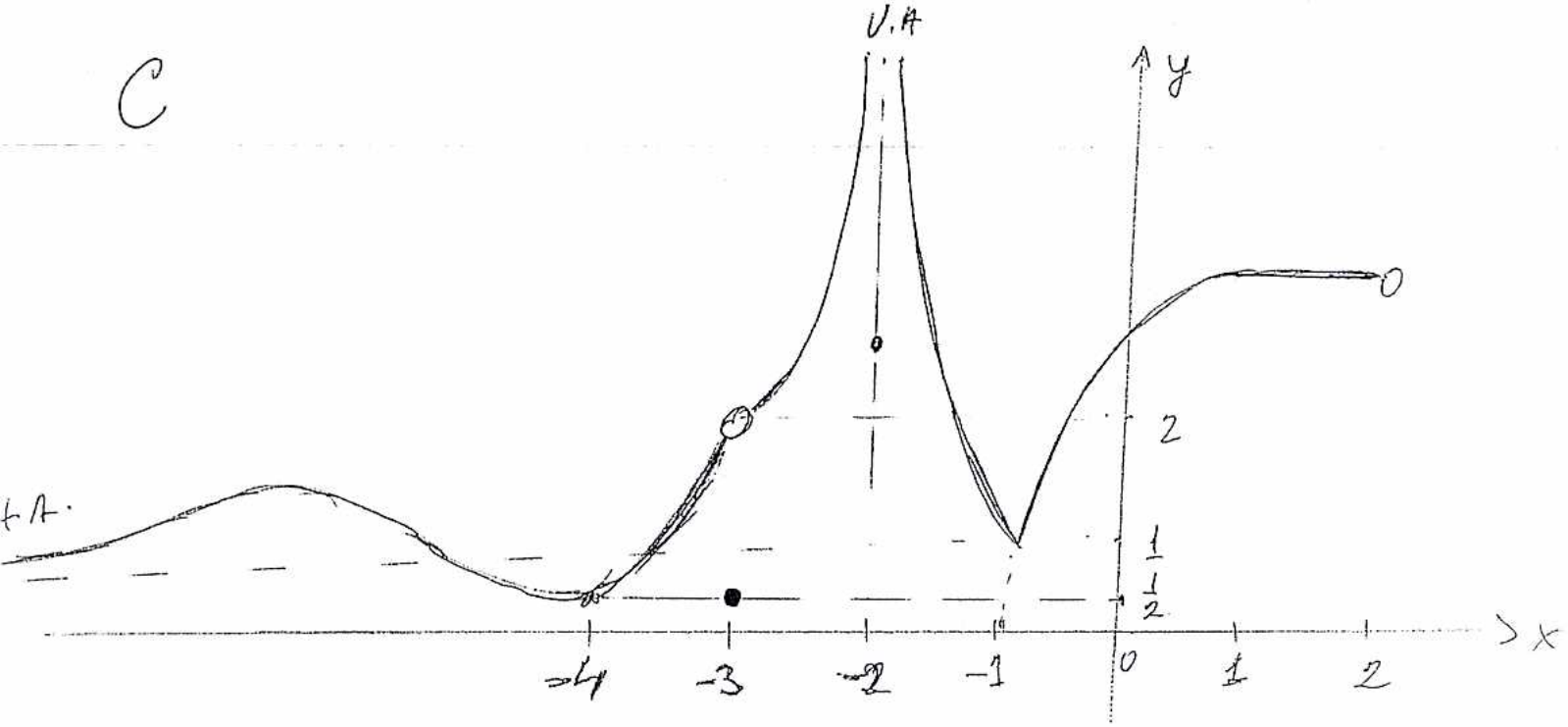
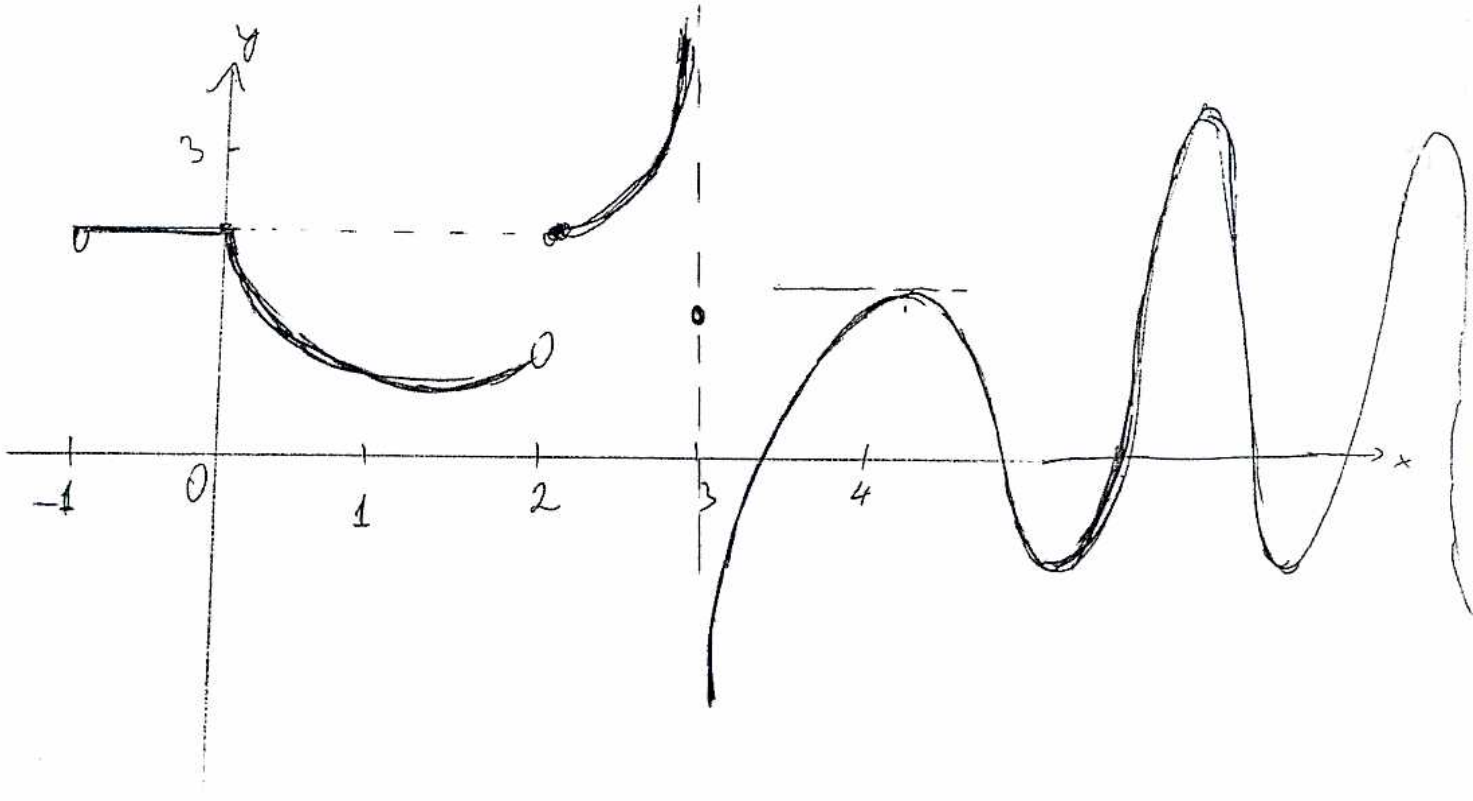
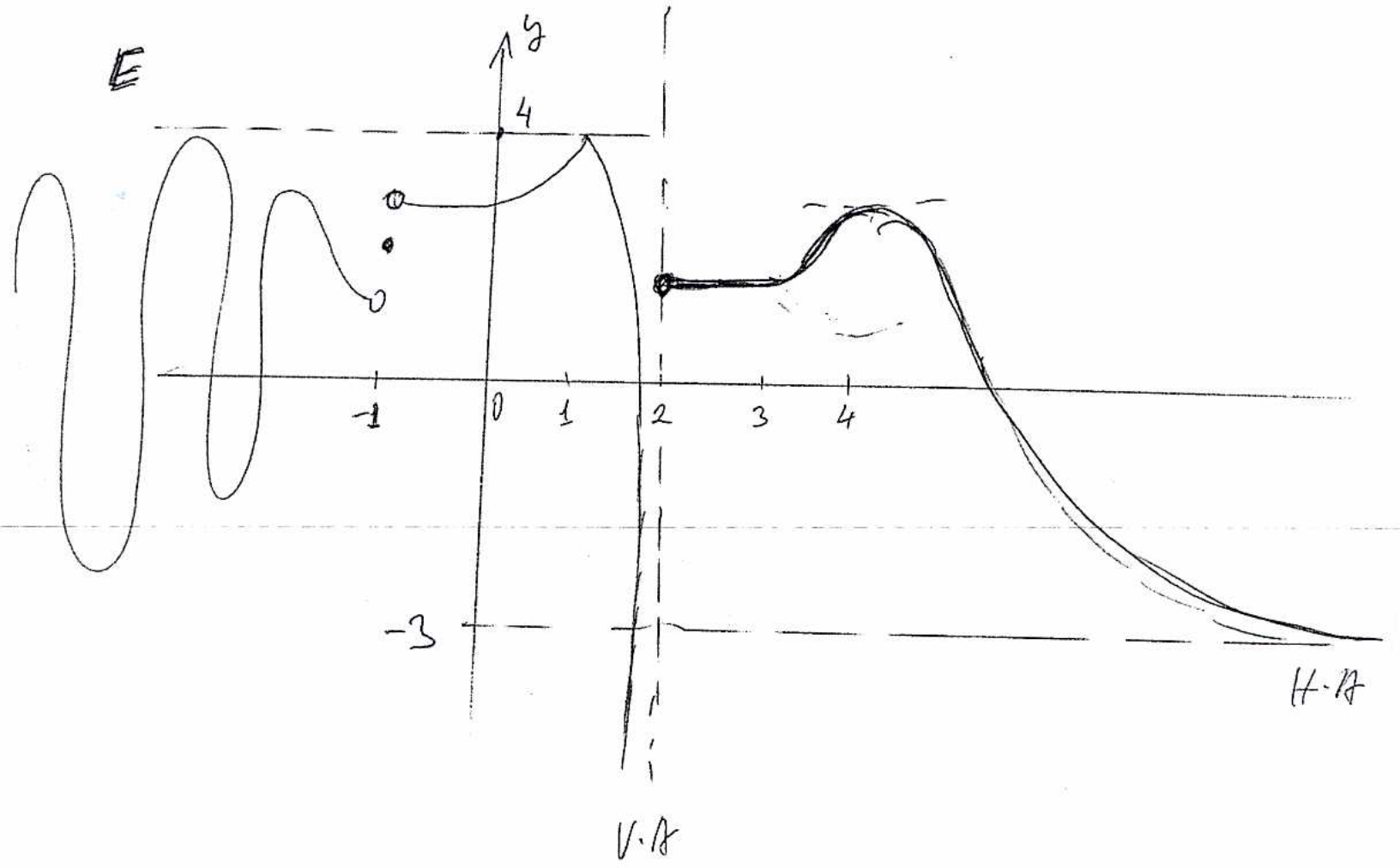


SCAN

D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