The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #2

Solution.

For 1a) For $f(x) = \dfrac{1}{1-x}\left(1 - \dfrac{4}{x+3}\right)$
the type of the limit is "$\frac{0}{0}$"as $x \to 1$ so we have to simplify
$f(x) = \dfrac{1}{1-x} \cdot \dfrac{x+3-4}{x+3} = \dfrac{-(1-x)}{(1-x)(x+3)} = \dfrac{-1}{x+3}$ for any $x \neq 1, -3$.
As $x \to 1$ the limit is $\frac{-1}{4}$.
For1b) as $x \to -3^+$ $x+3>0$
we can use the simplification from above and the type of the limit is "$\frac{-1}{0^+}$"
so the limit is $-\infty$. OR
from the original formula the limit is $L = \frac{1}{4} \cdot \left(1 - "\frac{4}{0^+}"\right) = -\infty$
For 1c) as $x \to +\infty$.
From the simplified formula the type is "$\frac{-1}{\infty}$" so the limit is 0.
OR
from the original the type is "$\frac{1}{-\infty}$" $\cdot \left(1 - "\frac{4}{\infty}"\right) = 0 \cdot (1-0) = 0$.
For 2)
For $f(x) = \sqrt{9-x^2}$ and $g(x) = \dfrac{3}{x-1}$
first the domains of the given functions for D_f solve $9 - x^2 \geq 0, (3-x)(3+x) \geq 0$
parabola open down, above the x-axis between roots OR
split points are $x = \pm 3$, testing $- -^{neg} - -_{-3} - - -^{pos} - - -_{3} - -^{neg} - - -$
so $D_f = [-3,3]$ $D_g = \{x \neq 1\}$ since $x - 1 \neq 0$.
$g \circ g(x) = g(g(x)) = \dfrac{3}{(...)-1} = \dfrac{3}{\frac{3}{x-1} - 1} = \dfrac{3}{\frac{3-(x-1)}{x-1}} = \dfrac{3(x-1)}{4-x}$
we must start in D_g i.e. $x \neq 1$ and we have to guarantee that
$4 - x \neq 0$ so $x \neq 4$ $D_{g \circ g} = \{x \neq 1 \wedge x \neq 4\}$
$f \circ g(x) = \sqrt{9-(..)^2} = \sqrt{9 - \frac{9}{(x-1)^2}} = \sqrt{9 \cdot \left(1 - \frac{1}{(x-1)^2}\right)} = 3 \cdot \sqrt{\frac{x^2-2x+1-1}{(x-1)^2}} = 3\sqrt{\frac{x(x-2)}{(x-1)^2}}$
we must start in $D_g, x \neq 1$ and quarantee that $\dfrac{x(x-2)}{(x-1)^2} \geq 0$,split points are $x = 0, 2, 1$
testing $- -^{pos} -_0 - - neg - -_1 - -^{neg} - -_2 -^{pos} - - -$so $D_{f \circ g} = (-\infty, 0] \cup [2, \infty)$.
For 3)
For $g(x) = \dfrac{4}{2x-8}$ and $f(x) = \sqrt{x^2 - 9}$
$D_g = \{x \neq 4\}, D_f = (-\infty, -3] \cup [3, +\infty)$ since we have to solve: $x^2 - 9 \geq 0$
$(x-3)(x+3) \geq 0$ parabola open up with roots $x = \pm 3$
OR $x^2 \geq 9$ $\sqrt{x^2} = |x| \geq 3$
Now, $g \circ g(x) = \dfrac{4}{2() - 8} = \dfrac{2 \cdot 2}{2\left[\left(\frac{4}{2x-8}\right) - 4\right]} = \dfrac{2}{\frac{4-8x+32}{2x-8}} = 2 \cdot \dfrac{2x-8}{36-8x} = \dfrac{4(x-4)}{4(9-2x)} = \dfrac{x-4}{9-2x}$
for $x \neq 4$ and $x \neq \frac{9}{2}$,so $D_{g \circ g} = (-\infty, 4) \cup (4, 4.5) \cup (4.5, +\infty)$.

For $g\circ f(x)=\dfrac{4}{2\,()-8}=\dfrac{4}{2\sqrt{x^2-9}-8}=$ you can simplify

$=\dfrac{2}{\sqrt{x^2-9}-4}\cdot\dfrac{\sqrt{x^2-9}+4}{\sqrt{x^2-9}+4}=\dfrac{2\left(\sqrt{x^2-9}+4\right)}{x^2-9-4^2}=\dfrac{2\left(\sqrt{x^2-9}+4\right)}{x^2-25}$

we know that $x\in D_f$ and that new denominator must be non-zero
$\sqrt{x^2-9}-4\neq 0, \sqrt{x^2-9}\neq 4$, so $x^2-9\neq 4^2, x^2\neq 25$
OR after simplification $x^2-25\neq 0$ i.e. $x\neq\pm 5$,together
$D_{g\circ f}=(-\infty,-5)\cup(-5,-3]\cup[3,5)\cup(5,+\infty)$. .

For 4a)

for $f(x)=\dfrac{1-4x^2}{6x^2-5x+1}$ as $x\to-\infty$ the type is "$\frac{-\infty}{\infty}$" so
divide top and bottom by x^2 :

$\lim\limits_{x\to-\infty}\dfrac{\frac{1}{x^2}-4}{6-\frac{5}{x}+\frac{1}{x^2}}=\dfrac{0-4}{6-0+0}=-\frac{4}{6}=-\dfrac{2}{3}$ (since "$\frac{1}{\pm\infty}$" $=0$)

For 4b)

as $x\to\frac{1}{2}$ the type is "$\frac{0}{0}$" and we have polynomials so factorize

$\lim\limits_{x\to\frac{1}{2}}\dfrac{(1-2x)(1+2x)}{(2x-1)(3x-1)}=\lim\limits_{x\to\frac{1}{2}}\dfrac{-(1+2x)}{3x-1}=\frac{-2}{\frac{1}{2}}=-4.$

For 4c) as $x\to\frac{1}{3}^-$

we can use the simplification from above but the type is "$\frac{neg\#}{0^-}$" since $x<\frac{1}{3}$ so $3x-1<0$

$\lim\limits_{x\to\frac{1}{3}^-}\dfrac{-(1+2x)}{3x-1}=$ "$\frac{-\frac{5}{3}}{0^-}$" $=$ "$\frac{1}{0^+}$" $=+\infty$

For 5a)

For $\dfrac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}$ as $x\to 3^+$

$\lim\limits_{x\to3^+}\dfrac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}=$ "$\frac{0}{0}$" $=\lim\limits_{x\to3^+}\dfrac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}\cdot\dfrac{\sqrt{3x}+3}{\sqrt{3x}+3}=\lim\limits_{x\to3^+}\dfrac{3x-3^2}{\sqrt{2x^2-6x}}\cdot\dfrac{1}{\sqrt{3x}+3}=$

$=\lim\limits_{x\to3^+}\dfrac{3(x-3)}{\sqrt{2x(x-3)}}\cdot\dfrac{1}{\sqrt{3x}+3}=\lim\limits_{x\to3^+}\dfrac{3}{\sqrt{2x}}\cdot\dfrac{x-3}{\sqrt{x-3}}\cdot\dfrac{1}{\sqrt{3x}+3}=$

$=\lim\limits_{x\to3^+}\dfrac{3}{\sqrt{2x}}\cdot\sqrt{x-3}\cdot\dfrac{1}{\sqrt{3x}+3}=\frac{3}{\sqrt{6}}\cdot 0\cdot\frac{1}{6}=0.$

For 5b) as $x\to+\infty$,

the type is "$\frac{\infty}{\infty}$" so we have to divide by the highest power in the denominator
in the original from by $x=\sqrt{x^2}$ $(x>0)$:

$\lim\limits_{x\to+\infty}\dfrac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}\cdot\dfrac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{\frac{3}{x}}-\frac{3}{x}}{\sqrt{2-\frac{6}{x}}}=\frac{0-0}{\sqrt{2}}=0$

OR we can used the simplified expresion from a)

$\lim\limits_{x\to+\infty}\dfrac{3}{\sqrt{2x}}\cdot\sqrt{x-3}\cdot\dfrac{1}{\sqrt{3x}+3}=\lim\limits_{x\to+\infty}3\sqrt{\frac{x-3}{2x}}\cdot\frac{1}{\sqrt{3x}+3}=$

$=\lim\limits_{x\to+\infty}3\sqrt{\frac{1}{2}-\frac{3}{2x}}\cdot\lim\limits_{x\to+\infty}\frac{1}{\sqrt{3x}+3}=\frac{3}{\sqrt{2}}\cdot 0=0$ since "$\frac{1}{\infty}$" $=0$.

For 5c) as $x\to 0$

the limit DNE (does not exist neither as a number nor as $\pm\infty$)
since the function is not defined for small negative x $(\sqrt{neg})$

For 6)

For $f(x) = \dfrac{\sqrt{3-x}}{x^2-4x+3}$

For 6 a) as $x \to 3^-$

the type is "$\frac{0}{0}$" and the function is defined for $x < 3$ and $x \neq 1$ we can simplify

$$f(x) = \frac{\sqrt{3-x}}{x^2-4x+3} = \frac{\sqrt{3-x}}{(x-3)(x-1)} = \frac{\sqrt{3-x}}{-(3-x)(x-1)} = \frac{\sqrt{3-x}}{-\sqrt{3-x}\sqrt{3-x}(x-1)} = -\frac{1}{\sqrt{3-x}\,(x-1)}$$

Now the type is "$\frac{-1}{0^+\cdot(2)}$" = "$\frac{1}{0^{--}}$" and the limit is $-\infty$.

For 6b) as $x \to 1^+$

We can use the simplification from above or at least identify the type "$\frac{\sqrt{2}}{0}$" so we have to investigate the sign of the bottom

Since $x > 1$ and $f(x) = \dfrac{\sqrt{3-x}}{(x-3)(x-1)}$ we can see that the type is :"$\frac{\sqrt{2}}{(-2)\cdot 0^+}$" = "$\frac{1}{0^-}$" and the limit is $-\infty$.

For 6c) as $x \to +\infty$.

the limit DNE (does not exists) since the function is not defined for big positive x .

For 7)

For $g(x) = \sqrt{3+x}$ and $f(x) = \sqrt{x-5}$

first the domains of the given functions $D_g = [-3, +\infty)$ since $3 + x \geq 0$;

$D_f = [5, +\infty)$ since $x - 5 \geq 0$.

$f \circ g(x) = f(g(x)) = \sqrt{(..) - 5} = \sqrt{\sqrt{3+x} - 5}$

we must start in D_g i.e. $x \in [-3, +\infty)$ and we have to guarantee that

$\sqrt{3+x} - 5 \geq 0$,so $\sqrt{3+x} \geq 5$

both sides are positive so we can square $(3 + x) \geq 25$, and $x \geq 22$,together

$D_{f\circ g} = [22, +\infty)$

$g \circ g(x) = \sqrt{3 + (..)} = \sqrt{3 + \sqrt{3+x}}$

we must start in $D_g = [-3, +\infty)$ and quarantee that $3 + \sqrt{3+x} \geq 0$

but it is always true for any $x \in [-3, +\infty)$ so $D_{g\circ g} = [-3, +\infty)$.

For 8a) as $x \to 0$

the type is "$\frac{0}{0}$" and if x is a small # ,neg. or pos,$x - 3$ is close to -3 so negative and $|x-3| = -(x-3) = 3-x$ $x+3$ is close to 3 so $x+3$ is positive and $|x+3| = x+3$ therefore

$$f(x) = \frac{|x-3| - |x+3|}{x} = \frac{3 - x - (x+3)}{x} = \frac{-2x}{x} = -2 \text{ for } x \neq 0$$

so the limit is -2.

ALSO $f(x) =$

$$\frac{|x-3| - |x+3|}{x} \cdot \frac{|x-3| + |x+3|}{|x-3| + |x+3|} = \frac{|x-3|^2 - |x+3|^2}{x \cdot (|x-3| + |x+3|)} = \frac{(x-3)^2 - (x+3)^2}{x \cdot (|x-3| + |x+3|)} =$$

$$= \frac{x^2 - 6x + 3^2 - (x^2 + 6x + 3^2)}{x \cdot (|x-3| + |x+3|)} = \frac{-12x}{x \cdot (|x-3| + |x+3|)} = \frac{-12}{(|x-3| + |x+3|)} \text{ for any } x \neq 0$$

Now the limit is $L = \frac{-12}{3+3} = -2$.

For8b) as $x \to -\infty$

For x big negative number $x-3$ is negative and $|x-3|=-(x-3)=3-x$,also $x+3$ is negative and

$|x+3|=-(x+3)=-3-x$,

$f(x)=\dfrac{|x-3|-|x+3|}{x}=\dfrac{3-x+x+3}{x}=\dfrac{6}{x}$ so the type of the limit is "$\frac{1}{\infty}$" and $L=0$.

ALSO for both b) and c)

using the siplification from above $f(x)=\dfrac{-12}{(|x-3|+|x+3|)}$ so the type is "$\frac{-12}{\infty}$" and the limit is 0.

For 8 c) as $x\to+\infty$.

For x big positive both expressions $x-3$ and $x+3$ are positive so we can ignore absolute values and

$f(x)=\dfrac{x-3-(x+3)}{x}=\frac{-6}{x}$ and the type of the limit is "$\frac{-6}{\infty}$" and the limit is 0.

For 9)

For $g(x)=\sqrt{3-x}$ and $f(x)=\dfrac{6}{3x-1}$

first the domains of the given functions $D_f=\left\{x\neq\frac{1}{3}\right\}$ since $3x-1\neq0$;

$D_g=(-\infty,3]$ since $3-x\geq0$.

$f\circ f(x)=f(f(x))=\dfrac{6}{3(...)-1}=\dfrac{6}{3\cdot\frac{6}{3x-1}-1}=\dfrac{6}{\frac{18-(3x-1)}{3x-1}}=\dfrac{6\,(3x-1)}{19-3x}$

we must start in D_f i.e. $x\neq\frac{1}{3}$ and we have to guarantee that $19-3x\neq0$ so $x\neq\frac{19}{3}$ and $D_{f\circ f}=\left\{x\neq\frac{1}{3}\wedge x\neq\frac{19}{3}\right\}$

$g\circ f(x)=\sqrt{3-(..)}=\sqrt{3-\frac{6}{3x-1}}=\sqrt{\frac{3(3x-1)-6}{3x-1}}=\sqrt{\frac{9x-9}{3x-1}}=3\sqrt{\frac{x-1}{3x-1}}$

we must start in $D_f, x\neq\frac{1}{3}$ and quarantee that $\dfrac{x-1}{3x-1}\geq0$,split points are $x=1,\frac{1}{3}$

testing $-^{pos}--_{\frac{1}{3}}--^{neg}--_{1}--^{pos}-$

so the domain is $D_{g\circ f}=\left(-\infty,\frac{1}{3}\right)\cup[1,+\infty)$