

**The University of Calgary**  
**Department of Mathematics and Statistics**  
**MATH 249**  
**Worksheet #5**

1. How much money one has to invest today at the interest of 3% compounded quarterly to get \$10,000 in 10 years?
2. Find the domain and derivative of  $f(x) = (3x)^\pi + \pi^{3x} + (\pi x)^{3x}$ .
3. Find  $y'$  if  $y = 2^{x^4} + \frac{2}{x^4} + \left(\frac{1}{x}\right)^x$ , for  $x > 0$ .
4. How long does it take to double your investment if the interest of 7 % is compounded  
 (a) yearly?      (b) monthly?
5. In the first few weeks after birth, a baby gains weight at a rate proportional to its weight. A baby weighing 4kg at birth weighs 4.4kg after 2 weeks.  
 How much did it weigh 4 days after birth ?
6. For  $f(x) = 3^x \ln \frac{3}{x}$  find  $f'(3)$ .
7. Find  $y'$  if  $y = x^{x^2} + \ln \frac{1}{1-x}$ , for  $0 < x < 1$ .
8. After 3 days a sample of radon-222 decayed to 58% of its original amount. What is half-life of radon-222?
9. Evaluate      (a)  $\lim_{x \rightarrow +\infty} \frac{x}{2^x - 1}$       (b)  $\lim_{x \rightarrow -\infty} \frac{x}{2^x - 1}$       (c)  $\lim_{x \rightarrow 0} \frac{x}{2^x - 1}$
10. Evaluate  
 (a)  $\lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}}$       (b)  $\lim_{x \rightarrow -\infty} \frac{x^2}{e^{3x}}$       (c)  $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x}$       (d)  $\lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{x}$ .
11. Solve for  $x$  :       $\frac{1}{2^{x+1}} = \frac{5}{4^x}$ .
12. Solve for  $x$ :      (a)  $\frac{1}{2} \ln(x + 3) + 1 = 0$ ;      (b)  $3^{x^2} = 9^{x-3}$ .

**SOLUTION**

**For1)**

the formula to use is  $A(t) = A_0 \left(1 + \frac{p}{100n}\right)^{nt}$  where  $t = 10, n = 4, p = 3$   
 $A = 10,000$        $A_0 = ?$       is the initial amount to invest      so

$10000 = A_0(1 + \frac{3}{400})^{40} = A_0(\frac{403}{400})^{40}$  multiply by reciprocal to isolate  $A_0$

$$A_0 = 10000(\frac{400}{403})^{40} = \$7416.48$$

**For 2)**  $x > 0$

$$f(x) = (3x)^\pi + \pi^{3x} + (\pi x)^{3x} = 3^\pi x^\pi + (\pi^3)^x + e^{3x \ln \pi x}$$

power + exp.f +  $e^u$  so by Chain Rule  $(e^u)' = e^u u'$

$$\begin{aligned} f'(x) &= 3^\pi (x^\pi)' + [(\pi^3)^x]' + e^{3x \ln \pi x} [3x \ln(\pi x)]' \text{ (Pr.R.)} \\ &= \pi 3^\pi x^{\pi-1} + (\pi^3)^x \ln \pi^3 + e^{3x \ln \pi x} \left[ 3 \ln \pi x + 3x \cdot \frac{1}{\pi x} \cdot \pi \right] = \\ &= \pi 3^\pi x^{\pi-1} + 3(\pi^3)^x \ln \pi + e^{3x \ln \pi x} [3 \ln \pi x + 3] \end{aligned}$$

ALSO change all terms into  $e^u$  and then Chain rule

$$f(x) = (3x)^\pi + \pi^{3x} + (\pi x)^{3x} = e^{\pi \ln 3x} + e^{3x \ln \pi} + e^{3x \ln \pi x}$$

so

$$\begin{aligned} f'(x) &= e^{\pi \ln 3x} (\pi \ln 3x)' + e^{3x \ln \pi} (3x \ln \pi)' + e^{3x \ln \pi x} (3x \ln \pi x)' = \\ &= e^{\pi \ln 3x} \left( \pi \cdot \frac{3}{3x} \right) + e^{3x \ln \pi} (3 \ln \pi) + e^{3x \ln \pi x} \left[ 3 \ln \pi x + 3x \cdot \frac{1}{\pi x} \cdot \pi \right] = \\ &= \frac{\pi}{x} (3x)^\pi + (3 \ln \pi) \pi^{3x} + [3 \ln \pi x + 3] (\pi x)^{3x} \end{aligned}$$

**For 3)**

$$\begin{aligned} y' &= (e^{x^4 \ln 2})' + (2x^{-4})' + (e^{x \ln \frac{1}{x}})' = e^{x^4 \ln 2} (x^4 \ln 2)' - 8x^{-5} + e^{-x \ln x} \cdot (-x \ln x)' = \\ &= 4x^3 \ln 2 e^{x^4 \ln 2} - 8x^{-5} - e^{-x \ln x} \left( 1 \cdot \ln x + x \cdot \frac{1}{x} \right) = 4x^3 \ln 2 e^{x^4 \ln 2} - 8x^{-5} - e^{-x \ln x} (\ln x + 1) \end{aligned}$$

Or by log.diff.BUT only for the last term  $u = \left(\frac{1}{x}\right)^x = e^{x \ln \frac{1}{x}}$  so

$$\ln u = x \ln \frac{1}{x} = x \ln x^{-1} = -x \ln x$$

$$\text{and } \frac{1}{u} \cdot u' = - \left( 1 \cdot \ln x + x \cdot \frac{1}{x} \right) = -(\ln x + 1) \text{ so } u' = - \left(\frac{1}{x}\right)^x (\ln x + 1)$$

for the first term we can use  $(2^u)' = 2^u \cdot \ln 2 \cdot u'$  so  $(2^{x^4})' = 2^{x^4} \cdot \ln 2 \cdot 4x^3$

and the second term is just a power so  $(2x^{-4})' = -8x^{-5}$ .

Together

$$y' = 4x^3 \ln 2 \cdot 2^{x^4} - 8x^{-5} - \left(\frac{1}{x}\right)^x (\ln x + 1) \quad \text{as above.}$$

**For 4a)**

we have to solve for  $t$   $2A_0 = A_0 \left(1 + \frac{7}{100}\right)^t$  cancel  $A_0$

thus  $2 = (1.07)^t$  take ln of both sides:

$$\ln 2 = t \ln 1.07, \text{ so } t = \frac{\ln 2}{\ln 1.07} = 10.24$$

We need 10 years and almost 3 months.

**For 4b)** we have to solve for  $t$   $2A_0 = A_0 \left(1 + \frac{7}{1200}\right)^{12t}$

as above thus  $2 = \left(\frac{1207}{1200}\right)^{12t}$  and

$$\ln 2 = 12t \ln \frac{1207}{1200}, \text{ and } t = \frac{\ln 2}{12(0.0058163)} = 9.929.$$

So this time we need less than 10 years.

**For 5)**

for the weight after  $t$  weeks  $W(t) = 4e^{kt}$ , given  $W(2) = 4.4 = 4e^{2k}$

solve for  $k$ :  $\frac{4.4}{4} = e^{2k}$ , apply ln to both sides:  $\ln 1.1 = 2k$ , so  $k = \frac{\ln 1.1}{2}$

Now, after 4 days means  $\frac{4}{7}$  of a week, so  $W = 4e^{k \frac{4}{7}} = 4e^{\frac{2}{7} \cdot \ln 1.1} = 4.11$  kg.

OR  $W(t) = 4e^{kt}$  where we measure time  $t$  in **days** then 2 weeks is 14 days  
 $W(14) = 4.4 = 4e^{14k}$  then  $k = \frac{\ln 1.1}{14}$   
 and for  $t = 4$   $W = 4e^{k4} = 4e^{4 \frac{\ln 1.1}{14}} = (\text{as above}) = 4.11$  kg

**For 6)**

first simplify  $f(x) = 3^x \cdot (\ln 3 - \ln x)$ , then use Product Rule

$$f'(x) = (3^x)' (\ln 3 - \ln x) + 3^x (\ln 3 - \ln x)' = 3^x \ln 3 (\ln 3 - \ln x) + 3^x \left(\frac{-1}{x}\right)$$

since  $(\ln 3)' = 0$  then  $x = 3$

$$\text{and } f'(3) = 3^3 \ln 3 (\ln 3 - \ln 3) + 3^3 \left(\frac{-1}{3}\right) = -9.$$

**For 7)**

simplify first

$$y = x^{x^2} + \ln \frac{1}{1-x} = e^{x^2 \ln x} - \ln(1-x) \quad y' = e^{x^2 \ln x} (x^2 \ln x)' - \frac{1}{1-x} (1-x)' =$$

$$= e^{x^2 \ln x} (2x \ln x + x) + \frac{1}{1-x}$$

OR you can use log.diff. but only for the first part

$$u = x^{x^2} \quad \ln u = x^2 \ln x \quad \frac{u'}{u} = 2x \ln x + x^2 \cdot \frac{1}{x} \quad u' = x^{x^2} (2x \ln x + x)$$

$$\text{and } y' = u' + \frac{1}{1-x}.$$

**For 8)**

the correct formula  $A(t) = A_0 e^{kt}$  where  $k < 0$ ,  $t$  in **days**,  $A_0 = 100\%$

$$\text{first info if } t = 3 \quad 58 = 100e^{3k} \text{ solve for } k \quad \ln \frac{58}{100} = 3k$$

$$k = \frac{\ln 0.58}{3} = -0.1815757$$

so  $A(t) = 100e^{kt}$  for  $k$  calculated above ;

now half-life  $T$  means we got 50%

$$50 = 100e^{kT} \quad \text{where } k = \frac{\ln 0.58}{3} \quad \frac{50}{100} = e^{kT}$$

$$\text{solve for } T \quad \ln 0.5 = kT \quad T = \frac{3 \ln 0.5}{\ln 0.58} = 3.8174 \text{ days}$$

**For 9) for a)**

$$\lim_{x \rightarrow +\infty} \frac{x}{2^x - 1} \quad \text{the type is } \frac{+\infty}{+\infty} \text{ so we can use L'Hop. rule}$$

$$\lim_{x \rightarrow +\infty} \frac{(x)'}{(2^x - 1)'} = \lim_{x \rightarrow +\infty} \frac{1}{2^x \ln 2} = \frac{1}{\infty} = 0$$

**for b)**

$$\lim_{x \rightarrow -\infty} \frac{x}{2^x - 1} \text{ since } 2^{-\infty} = 0 \text{ No L'H.R.} \quad \lim_{x \rightarrow -\infty} \frac{x}{2^x - 1} = \frac{-\infty}{-1} = +\infty.$$

**for c)**

$$\lim_{x \rightarrow 0} \frac{x}{2^x - 1} \text{ the type is } \frac{0}{0} \text{ so L'Hop. Rule again} \quad \lim_{x \rightarrow 0} \frac{x}{2^x - 1} = \lim_{x \rightarrow 0} \frac{1}{2^x \ln 2} = \frac{1}{\ln 2}$$

**For 10)**

**for a)**

the type is  $\frac{\infty}{\infty}$  so use L'Hop. Rule

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}} = \lim_{x \rightarrow \infty} \frac{2x}{3e^{3x}} \text{ (again)} = \lim_{x \rightarrow \infty} \frac{2}{9e^{3x}} = \frac{2}{\infty} = 0$$

for b)

the type is " $\frac{\infty}{0^+}$ " since " $e^{-\infty}$ " = 0

so  $\lim_{x \rightarrow -\infty} \frac{x^2}{e^{3x}} = +\infty(+\infty) = +\infty$  since " $\frac{1}{0^+}$ " =  $+\infty$ . No L'H.R.

also  $\lim_{x \rightarrow -\infty} \frac{x^2}{e^{3x}} = \lim_{x \rightarrow -\infty} x^2 e^{-3x} = +\infty \cdot e^{+\infty} = +\infty$

for c)  $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} = \frac{\infty}{\infty}$  (L'H.R.) =  $\lim_{x \rightarrow \infty} \frac{2(\ln x) \frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{2 \ln x}{x} =$

again L'H.R. =  $\lim_{x \rightarrow \infty} \frac{\frac{2}{x}}{1} = \frac{2}{\infty} = 0$

for d)  $\lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{x} = \frac{\infty}{0^+} = \lim_{x \rightarrow 0^+} (\ln x)^2 \cdot \frac{1}{x} = (-\infty)^2 (+\infty) = +\infty$   
(No L'H.R.)

**For 11)**

cross multiply first, so  $4^x = 5 \cdot 2^{x+1}$ , then apply ln to both sides

$$\ln 4^x = \ln(5 \cdot 2^{x+1}) = \ln 5 + \ln 2^{x+1}$$

$$\text{thus } x \ln 4 = \ln 5 + (x+1) \ln 2 \quad x \ln 4 = \ln 5 + x \ln 2 + \ln 2$$

$$\text{and } x \ln 4 - x \ln 2 = \ln 5 + \ln 2$$

$$\text{So } x(\ln 4 - \ln 2) = \ln(5 \cdot 2) \text{ and finally } x \ln \frac{4}{2} = \ln 10 \quad x = \frac{\ln 10}{\ln 2}.$$

**For 12a)**

$$\frac{1}{2} \ln(x+3) = -1 \quad \ln(x+3) = -2$$

then exp.f. to both sides and  $(x+3) = e^{-2}$

and so  $x = e^{-2} - 3$ .

**b)**

Take log of both sides:  $\ln 3^{x^2} = \ln 9^{x-3}$

$$x^2 \ln 3 = (x-3) \ln 9 = (x-3) \ln 3^2 = (x-3) \cdot 2 \ln 3$$

$$\text{cancel } \ln 3 \text{ and } x^2 = 2(x-3) = 2x - 6$$

everything on one side:  $x^2 - 2x + 6 = 0$ , discriminant of this polynomial is

$D = (-2)^2 - 4 \cdot 1 \cdot 6 = -20$  so no real roots exist and the problem has **NO solution**.

Also we can change both sides to the same base:  $3^{x^2} = (3^2)^{x-3} = 3^{2x-6}$

and by comparing the exponents we get the same quadratic equation.