Practice Problems S2

1. If $f(x) = \frac{1+x}{1-x}$ and $g(x) = \frac{1}{x}$, find formulas for the functions $f+g$, $f-g$, $f\,g$, $\dfrac{f}{g}$, $\dfrac{g}{f}$, $f \circ g$, $g \circ f$, and specify their domains.

2. Express the following functions as compositions of two functions:

   (a) $h(x) = \sqrt{1 - \sqrt[3]{x}}$

   (b) $h(x) = \dfrac{\sqrt{x} - 3}{x + 5\sqrt{x} + 6}$

3. Check that the following functions are one-to-one. Find the inverses and specify their domains and ranges.

   (a) $f(x) = \dfrac{x}{1+x}$

   (b) $f(x) = (2x-1)^3 - 8$

4. Simplify the following expressions

   (a) $\dfrac{(a^5)^2 a^{-6}}{\sqrt[3]{a^2}}$

   (b) $(1/3)^x 9^{x/2}$

   (c) $x^{1/\log_a x}$

   (d) $e^{3\ln 2 - 4\ln 3}$

   (e) $2\log_3 12 - 4\log_3 6$

5. Solve the following equations for x:

   (a) $\log_7(x^{\frac{3}{2}}) = 2 - \log_7 \sqrt{x}$;

   (b) $2^{4-x^2} = \frac{1}{8^x}$;

(c) $e^{2x} - e^x - 6 = 0$;

(d) $\ln(x) + \ln(x-1) = \ln 6$.

6. Evaluate the following limits if they exit or are infinity. Explain your answers.

   (a) $\lim_{x \to -2} (3x^2 - 2x + 8)$;

   (b) $\lim_{x \to \frac{3}{2}} \dfrac{2x-3}{|2x-3|}$;

   (c) $\lim_{x \to 0} \dfrac{1}{x\sqrt{x+1}} - \dfrac{1}{x}$;

   (d) $\lim_{x \to 2} \dfrac{4x-8}{\sqrt{2x+5} - \sqrt{x^2+5}}$;

   (e) $\lim_{x \to -2^-} \dfrac{x^2+2x}{x^2-4}$;

   (f) $\lim_{x \to -2^+} \dfrac{x^2-4}{x(x+2)^2}$.

Solutions

1. The functions $f(x) = \frac{1+x}{1-x}$ and $g(x) = \frac{1}{x}$ have domains $D_f = \mathbb{R} \setminus \{1\}$ and $D_g = \mathbb{R}\setminus\{0\}$. $(f+g)(x) = \frac{1+x}{1-x} + \frac{1}{x} = \frac{1+x^2}{x(1-x)}$, $(f-g)(x) = \frac{1+x}{1-x} - \frac{1}{x} = \frac{x^2+2x-1}{x(1-x)}$, $(fg)(x) = \frac{1+x}{x(1-x)}$, $(f/g)(x) = \frac{x(1+x)}{1-x}$ and $(f \circ g)(x) = \frac{1+x}{x-1}$ are defined on $D_f \cap D_g = \mathbb{R} \setminus \{0, 1\}$; and $(g \circ f)(x) = \frac{1-x}{1+x}$ has domain $\mathbb{R} \setminus \{-1, 1\}$ and $(g/f)(x) = \frac{1-x}{x(1+x)}$ is defined on $\mathbb{R} \setminus \{-1, 0, 1\}$.

2. $h(x) = f \circ g(x)$, where
   (a) $f(x) = \sqrt{1-x}$, $g(x) = \sqrt[3]{x}$; or $f(x) = \sqrt{x}$, $g(x) = 1 - \sqrt[3]{x}$;
   (b) $f(x) = \frac{x-3}{x^2+5x+6}$, $g(x) = \sqrt{x}$.

3. (a) $f(x) = \frac{x}{1+x}$ has domain $D_f = \mathbb{R} \setminus \{-1\}$. For $x, y \in D_f$, if $f(x) = f(y)$, i.e., $\frac{x}{1+x} = \frac{y}{1+y}$, then $x(1+y) = y(1+x)$ which implies that $x = y$. This shows that $f(x)$ is one-to-one. Therefore, $f(x)$ is invertible. Its inverse f^{-1} has range, the domain of f, i.e., $\text{Range}(f^{-1}) = D_f = \mathbb{R} \setminus \{-1\}$. To find the inverse, solve the equation $y = f(x)$, i.e., $y = \frac{x}{1+x}$ for x to get $x = f^{-1}(y) = \frac{y}{1-y}$. So, $f^{-1}(x) = \frac{x}{1-x}$ with domain $D_{f^{-1}} = \mathbb{R} \setminus \{1\}$. Finally, $\text{Range}(f) = D_{f^{-1}} = \mathbb{R} \setminus \{1\}$.
   (b) $(2x-1)^3 - 8 = (2y-1)^3 - 8 \Rightarrow (2x-1)^3 = (2y-1)^3 \Rightarrow 2x - 1 = 2y - 1 \Rightarrow x = y$. This proves that $f(x) = (2x-1)^3 - 8$ is one-to-one. Solve $y = f(x) = (2x-1)^3 - 8$ for x to find the inverse $f^{-1}(x) = \frac{1}{2} + \frac{1}{2}\sqrt[3]{y+8}$. $D_f = R_{f^{-1}} = \mathbb{R} = D_{f^{-1}} = R_f$.

4. (a) $\frac{(a^5)^2 a^{-6}}{\sqrt[3]{a^2}} = \frac{a^{10} a^{-6}}{a^{2/3}} = a^{10-6-2/3} = a^{10/3}$;
   (b) $(\frac{1}{3})^x 9^{x/2} = \frac{3^{2x/2}}{3^x} = 1$;
   (c) Since $\frac{1}{\log_a x} = \frac{\log_a a}{\log_a x} = \log_x a$, we have $x^{\frac{1}{\log_a x}} = x^{\log_x a} = a$;
   (d) $e^{3\ln 2 - 4\ln 3} = e^{\ln 2^3 - \ln 3^4} = e^{\ln(2^3/3^4)} = 2^3/3^4 = 8/81$;

(e) $2\log_3 12 - 4\log_3 6 = \log_3 12^2 - 4\log_3 6^4 = \log_3 12^2/6^4 = \log_3 4^2 \times 3^2/(2^4 \times 3^4) = \log_3 1/3^2 = -2$.

5. (a) $\log_7(x^{\frac{3}{2}}) = 2 - \log_7 \sqrt{x} \iff \log_7(x^{\frac{3}{2}}) = \log_7 7^2 - \log_7 \sqrt{x} = \log_7 7^2/x^{\frac{1}{2}}$. So, $x^{\frac{3}{2}} = 7^2/x^{\frac{1}{2}} \Longrightarrow x = 7$.

(b $2^{4-x^2} = \frac{1}{8^x} \iff 2^{4-x^2} = 2^{-3x} \iff 4 - x^2 = -3x \iff x = 4 \quad \text{or} \quad x = -1$

(c) Set $u = e^x$. Then $u^2 - u - 6 = 0 \Longrightarrow u = 3$. So, $e^{2x} - e^x - 6 = 0$ has one solution $x = \ln 3$.

(d) $\ln(x) + \ln(x-1) = \ln 6 \Longrightarrow \ln x(x-1) = \ln 6 \Longrightarrow x(x-1) = 6 \Longrightarrow x = 3$.

6. (a) $\lim_{x\to -2}(3x^2 - 2x + 8) = 3 \times (-2)^2 - 2 \times (-2) + 8 = 12 + 4 + 8 = 24$;

(b) $\lim_{x\to \frac{3}{2}^-} \frac{2x-3}{|2x-3|} = \lim_{x\to \frac{3}{2}^-} \frac{2x-3}{-(2x-3)} = -1 \neq 1 = \lim_{x\to \frac{3}{2}^+} \frac{2x-3}{|2x-3|} = \lim_{x\to \frac{3}{2}^+} \frac{2x-3}{2x-3}$. Therefore, $\lim_{x\to \frac{3}{2}} \frac{2x-3}{|2x-3|}$ does not exist;

(c)

$$\begin{aligned}
& \lim_{x\to 0} \frac{1}{x\sqrt{x+1}} - \frac{1}{x} = \lim_{x\to 0} \frac{1-\sqrt{x+1}}{x\sqrt{x+1}} \\
= & \lim_{x\to 0} \frac{(1-\sqrt{x+1})(1+\sqrt{x+1})}{x\sqrt{x+1}(1+\sqrt{x+1})} = \lim_{x\to 0} \frac{1-x-1}{x\sqrt{x+1}(1+\sqrt{x+1})} \\
= & \lim_{x\to 0} \frac{-1}{\sqrt{x+1}(1+\sqrt{x+1})} = \frac{-1}{1+1} = -\frac{1}{2};
\end{aligned}$$

(d)

$$
\begin{aligned}
& \lim_{x\to 2} \frac{4x-8}{\sqrt{2x+5}-\sqrt{x^2+5}} \\
= \ & \lim_{x\to 2} \frac{(4x-8)(\sqrt{2x+5}+\sqrt{x^2+5})}{(\sqrt{2x+5}-\sqrt{x^2+5})(\sqrt{2x+5}+\sqrt{x^2+5})} \\
= \ & \lim_{x\to 2} \frac{(4x-8)(\sqrt{2x+5}+\sqrt{x^2+5})}{(2x+5)-(x^2+5)} \\
= \ & \lim_{x\to 2} \frac{(4x-8)(\sqrt{2x+5}+\sqrt{x^2+5})}{2x-x^2} \\
= \ & \lim_{x\to 2} \frac{4(x-2)(\sqrt{2x+5}+\sqrt{x^2+5})}{-x(x-2)} \\
= \ & \lim_{x\to 2} \frac{4(\sqrt{2x+5}+\sqrt{x^2+5})}{-x} = \frac{4(\sqrt{9}+\sqrt{9})}{-2} = \frac{24}{-2} = -12;
\end{aligned}
$$

(e) $\lim\limits_{x\to -2^-} \frac{x^2+2x}{x^2-4} = \lim\limits_{x\to -2^-} \frac{x(x+2)}{(x-2)(x+2)} = \lim\limits_{x\to -2^-} \frac{x}{x-2} = \frac{-2}{-2-2} = \frac{1}{2}$;

(f) $\lim\limits_{x\to -2^+} \frac{x^2-4}{x(x+2)^2} = \lim\limits_{x\to -2^+} \frac{(x-2)(x+2)}{x(x+2)^2} = \lim\limits_{x\to -2^+} \frac{x-2}{x(x+2)} = +\infty$.