Practice Problems S4

1. Use techniques of differentiation to find the derivatives of the following functions:

   (a) $f(x) = \frac{\sin(x)}{\cos(x)+x\sin(x)}$; (b) $f(x) = x^3 \tan^2(x^4)$; (c) $f(x) = \frac{1}{\sqrt{1+x^2}}$.

2. Find the (x, y)-coordinates of the points on the graph of the function $y = x^3$ where the tangent lines have slope $m = 3$.

3. Consider a curve described by the equation $x^2y - xy^2 = 2\cos(y - 1)$.

   (a) Use implicit differentiation to find $y' = \frac{dy}{dx}$.

   (b) Find the equation of the tangent line to the curve at $(-1, 1)$.

4. Find the tangent line to the parabola $x = y^2$ at $(4, -2)$.

5. (a) Find the local linear approximation of $f(x) = \sin x$ at π.

   (b) Estimate the value of $\sin \frac{3\pi}{4}$.

   (c) What is the error?

6. Let $f(x) = \sqrt[3]{x}$.

   (a) Find the local linear approximation of $f(x)$ at $x = 1$;

   (b) Use this approximation to estimate $\sqrt[3]{1.09}$.

7. Use a suitable local linear approximation to estimate the value of $\frac{1}{2.003}$.

Solutions

1. (a) $f'(x) = (\frac{\sin x}{\cos x + x \sin x})' = \frac{\cos^2 x}{(\cos x + x\sin x)^2}$ (quotient/product rules),

   (b) $f'(x) = (x^3 \tan^2(x^4))' = x^2 \tan(x^4)\left(3\tan(x^4) + 8x^4 \sec^2(x^4)\right)$ (product/chain rules),

   (c) $f'(x) = (\frac{1}{\sqrt{1+x^2}})' = -\frac{x}{(\sqrt{1+x^2})^3}$.

2. If (x_0, y_0) are the coordinates of a point on the graph of $f(x)$, then $y_0 = f(x_0)$. If the tangent line has slope $3 = m = f'(x_0) = \frac{df(x)}{dx}_{|x=x_0} = 3x_0^2$, then $x_0 = \pm 1$. So, $(1,1)$ and $(-1,-1)$.

3. (a)

$$\begin{aligned} \frac{d}{dx}(x^2y - xy^2) &= 2\frac{d}{dx}\cos(y-1) \\ 2xy + x^2y' - y^2 - 2xyy' &= -2y'\sin(y-1) \end{aligned}$$

   This implies that $y' = \frac{y^2 - 2xy}{x^2 - 2xy + 2\sin(y-1)}$.

   (b) The tangent line to the curve at the point $(-1,1)$ has slope $m = y'_{|x=-1,y=1} = \frac{1+2}{1+2+0} = 1$. Therefore, $y = (x+1)+1 = x+2$.

4. $\frac{d}{dx}x = \frac{d}{dx}y^2 \Longrightarrow 1 = 2yy'$. At $(4,-2)$, $y' = 1/(2(-2)) = -1/4$. The tangent line has equation $y = -\frac{1}{4}(x-4) - 2$.

5. (a) $f(x) = \sin(x) \Longrightarrow f(\pi) = \sin(\pi) = 0$, $f'(x) = (\sin(x))' = \cos(x) \Longrightarrow f'(\pi) = \cos(\pi) = -1$. So, $L(x) = f(\pi) + f'(\pi)(x-\pi)$, i.e., $L(x) = -x + \pi$ is the local linear approximation of $f(x) = \sin(x)$ at π.

   (b) $\sin(\frac{3\pi}{4}) = f(\frac{3\pi}{4}) \approx L(\frac{3\pi}{4}) = -\frac{3\pi}{4} + \pi = \frac{\pi}{4}$.

   (c) The error in this linear approximation is $E(\frac{3\pi}{4}) = |f(\frac{3\pi}{4}) - L(\frac{3\pi}{4})| = |\sin(\frac{3\pi}{4}) - \frac{\pi}{4}| = \frac{\pi}{4} - \frac{\sqrt{2}}{2}$.

6. (a) $L(x) = f(1) + f'(1)(x-1)$, where $f(1) = \sqrt[3]{1} = 1$ and $f'(1) = \frac{d}{dx}(\sqrt[3]{x})_{|x=1} = \frac{d}{dx}(x^{\frac{1}{3}})_{|x=1} = \frac{1}{3}x^{-\frac{2}{3}}_{|x=1} = \frac{1}{3}$, i.e., $L(x) = 1 + \frac{1}{3}(x-1)$.

   (b) $\sqrt[3]{1.09} = f(1.09) \approx L(1.09) = 1 + \frac{1}{3}(1.09 - 1) = 1.03$.

7. Take $f(x) = \frac{1}{x}$, $x_0 = 2$. So, $f(2) = 1/2$, $f'(x) = -1/x^2$, $f'(2) = -1/4$. The local linear approximation at $x = 2$ is $L(x) = f(2) + f'(2)(x-2)$, i.e., $L(x) = 1/2 - (x-2)/4$. Finally, $\frac{1}{2.003} = f(2.003) \approx L(2.003) = 0.49925$.