Practice Problems S5

1. Find the derivatives of the following functions:

   (a) $f(x) = \frac{1}{\ln(x+e^{-2x})}$;

   (b) $f(x) = \log_x(2x+3)$;

   (c) $f(x) = e^{x^2+2x+3}\ln(x+4)$.

2. Use logarithmic differentiation to find the derivatives of the following functions:

   (a) $f(x) = \frac{\sqrt{1+x}(1-x)^{\frac{1}{3}}}{(1+5x)^{\frac{4}{5}}}$;

   (b) $f(x) = x^{\cos x}$;

   (c) $f(x) = (\sec(x))^x$.

3. Evaluate the following limits using L'Hôpital's rule if possible:

   (a) $\lim_{x\to 0} \frac{xe^x}{1-e^x}$;

   (b) $\lim_{x\to 0} \frac{e^x-1}{\sin x}$;

   (c) $\lim_{x\to 1} \frac{\ln x}{x^2-1}$;

   (d) $\lim_{x\to 0} \frac{(1-x)e^x-1}{x\sin x}$;

   (e) $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{e^x-1})$.

4. Find all intervals where the following functions are increasing or decreasing, concave up or concave down:

   (a) $f(x) = x^3+3x^2+1$;

   (b) $f(x) = e^{-x^2/2}$.

   Does $f(x)$ have inflection points? Explain your answer.

Solutions

1. (a)

$$\begin{aligned} f'(x) &= \left(\frac{1}{\ln(x+e^{-2x})}\right)' = -\frac{(\ln(x+e^{-2x}))'}{\ln^2(x+e^{-2x})} \\ &= -\frac{1}{\ln^2(x+e^{-2x})}\cdot\frac{(x+e^{-2x})'}{x+e^{-2x}} \\ &= -\frac{1-2e^{-2x}}{x+e^{-2x}}\cdot\frac{1}{\ln^2(x+e^{-2x})}. \end{aligned}$$

(b) $f(x) = \log_x(2x+3) = \frac{\ln(2x+3)}{\ln x}$. So,

$$\begin{aligned} f'(x) &= \left(\frac{\ln(2x+3)}{\ln x}\right)' = \frac{\ln x(\ln(2x+3)' - (\ln x)'\ln(2x+3)}{\ln^2 x} \\ &= \frac{\frac{2\ln x}{2x+3} - \frac{\ln(2x+3)}{x}}{\ln^2 x} = \frac{2x\ln x - (2x+3)\ln(2x+3)}{x(2x+3)\ln^2 x}. \end{aligned}$$

(c)

$$\begin{aligned} \left(e^{x^2+2x+3}\ln(x+4)\right)' &= (e^{x^2+2x+3})'\ln(x+4) + e^{x^2+2x+3}(\ln(x+4))' \\ &= (x^2+2x+3)'e^{x^2+2x+3} + e^{x^2+2x+3}\frac{(x+4)'}{x+4} \\ &= (2x+2)3^{x^2+2x+3} + \frac{e^{x^2+2x+3}}{x+4}. \end{aligned}$$

2. (a)

$$\begin{aligned} \lim_{x\to 0}\frac{xe^x}{1-e^x} &= \text{„}\frac{0}{0}\text{„} \\ &= \lim_{x\to 0}\frac{(xe^x)'}{(1-e^x)'} \quad \text{(L'Hôpital's rule)} \\ &= \lim_{x\to 0}\frac{xe^x+e^x}{-e^x} = -1; \end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-1}{\sin x} &= \text{„}\frac{0}{0}\text{„} \\
&= \lim_{x\to 0}\frac{(e^x-1)'}{(\sin x)'} \quad \text{(L'Hôpital's rule)} \\
&= \lim_{x\to 0}\frac{e^x}{\cos x} = 1;
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to 1}\frac{\ln x}{x^2-1} &= \text{„}\frac{\ln 1}{1-1} = \frac{0}{0}\text{„} \\
&= \lim_{x\to 0}\frac{(\ln x)'}{(x^2-1)'} \quad \text{(L'Hôpital's rule)} \\
&= \lim_{x\to 0}\frac{1/x}{2x} = 1/2;
\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to 0}\frac{(1-x)e^x-1}{x\sin x} &= \text{„}\frac{0}{0}\text{„} \\
&= \lim_{x\to 0}\frac{[(1-x)e^x-1]'}{(x\sin x)'} \quad \text{(L'Hôpital's rule)} \\
&= \lim_{x\to 0}\frac{(1-x)e^x-e^x}{x\cos x+\sin x} = \lim_{x\to 0}\frac{-xe^x}{x\cos x+\sin x} = \text{„}\frac{0}{0}\text{„} \\
&= \lim_{x\to 0}\frac{(-xe^x)'}{(x\cos x+\sin x)'} \quad \text{(L'Hôpital's rule)} \\
&= \lim_{x\to 0}\frac{-e^x-xe^x}{\cos x-x\sin x+\cos x} = -1/2;
\end{aligned}$$

(e) $\lim_{x\to 0}(\frac{1}{x}-\frac{1}{e^x-1})$.

$$\lim_{x\to 0}(\frac{1}{x}-\frac{1}{e^x-1}) = \text{"}\infty-\infty\text{"}$$

$$\begin{aligned}
&= \lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} \\
&= \lim_{x\to 0} \frac{[e^x - 1 - x]'}{[x(e^x - 1)]'} \quad \text{(L'Hôpital's rule)} \\
&= \lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + xe^x} = \text{,,}\frac{0}{0}\text{''} \\
&= \lim_{x\to 0} \frac{(e^x - 1)'}{(e^x - 1 + xe^x)'} \quad \text{(L'Hôpital's rule)} \\
&= \lim_{x\to 0} \frac{e^x}{e^x + xe^x + e^x} = 1/2.
\end{aligned}$$

3. (a)

$$\begin{aligned}
\ln|f(x)| &= \ln\left|\frac{\sqrt{1+x}(1-x)^{\frac{1}{3}}}{(1+5x)^{\frac{4}{5}}}\right| \\
&= \frac{1}{2}\ln|1+x| + \frac{1}{3}\ln|1-x| - \frac{4}{5}\ln|1+5x|.
\end{aligned}$$

So,

$$\begin{aligned}
\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln|f(x)| &= \left(\frac{1}{2}\ln|1+x| + \frac{1}{3}\ln|1-x| - \frac{4}{5}\ln|1+5x|\right)' \\
&= \frac{1}{2(1+x)} - \frac{1}{3(1-x)} - \frac{4}{1+5x}.
\end{aligned}$$

Therefore,

$$f'(x) = \frac{\sqrt{1+x}(1-x)^{\frac{1}{3}}}{(1+5x)^{\frac{4}{5}}}\left(\frac{1}{2(1+x)} - \frac{1}{3(1-x)} - \frac{4}{1+5x}\right).$$

(b) $f(x) = x^{\cos x}$ is defined for $x > 0$. We have:

$$\ln f(x) = \ln(x^{\cos x}) = \cos x \ln x.$$

So,

$$\frac{f'(x)}{f(x)} = (\cos x \ln x)' = -\sin x \ln x + \frac{\cos x}{x}$$
$$\Longrightarrow f'(x) = (x^{\cos x})' = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$

(c) $\ln f(x) = \ln((\sec x)^x) = x \ln \sec x$. So,

$$\frac{f'(x)}{f(x)} = (x \ln \sec x)' = \ln \sec x + x\frac{\sec x \tan x}{\sec x} = \ln \sec x + x \tan x$$
$$\Longrightarrow f'(x) = ((\sec x)^x)' = \sec x)^x(\ln \sec x + x \tan x).$$

4. (a) $f'(x) = (x^3 + 3x + 1)' = 3x^2 + 6x = 3x(x + 2)$. $f(x)$ is increasing where $f'(x) > 0$, i.e., on $(-\infty, -2]$ and $[0, +\infty]$. $f(x)$ is decreasing where $f'(x) < 0$, i.e., on $[-2, 0]$.

   $f''(x) = (3x^2 + 6x)' = 6(x + 1)$. $f(x)$ is concave up where $f''(x) > 0$, i.e., on $(-1, +\infty)$. $f(x)$ is concave down where $f''(x) < 0$, i.e., on $(-\infty, -1)$. Since $f(x)$ change its concavity at -1, $(-1, f(-1)) = (-1, -3)$ is an inflection point.

   (b) The function $f(x) = e^{-x^2/2}$ has derivatives $f'(x) = -xe^{-x^2/2}$ and $f''(x) = (x^2 - 1)e^{-x^2/2}$. $f(x)$ is increasing where $f'(x) > 0$, i.e., on $(-\infty, 0]$, and decreasing where $f'(x) < 0$, i.e., $[0, +\infty)$. $f(x)$ is concave up where $f''(x) > 0$, i.e., on $(-\infty, -1)$ and $(1, +\infty)$, concave down where $f''(x) < 0$, i.e., on $(-1, 1)$. $f(x)$ has two inflection points $(-1, e^{-1/2})$ and $(1, e^{-1/2})$.