Practice Problems S6

1. Given

$$f(x) = \frac{x^2-9}{x^2-4}, \quad f'(x) = \frac{10x}{(x-2)^2(x+2)^2},$$
$$f''(x) = \frac{-10(3x^2+4)}{(x-2)^3(x+2)^3}, \quad \lim_{x\to-\infty} f(x) = 1 = \lim_{x\to+\infty} f(x),$$
$$\lim_{x\to-2^-} f(x) = -\infty = \lim_{x\to 2^+} f(x), \quad \lim_{x\to-2^+} f(x) = +\infty = \lim_{x\to 2^-} f(x),$$

sketch the graph of f (Highlight all asymptotes, intercepts, relative extrema and inflection points if any).

2. Let $f(x) = (x^2+x)^{\frac{2}{3}}$.
   (a) Find all critical and singular points of $f(x)$;
   (b) Find the absolute maximum and the absolute minimum values of $f(x)$ on $[-2, 3]$. Where does $f(x)$ attain these values (absolute minimum and absolute maximum points)?

3. Let $f(x) = x^3 - 3x^2 - 9x$.
   (a) Find all relative (local) extrema of $f(x)$ if any;
   (b) Find all intervals where $f(x)$ is concave up or concave down;
   (c) Find all inflection points if any.

4. A cylindrical can, open at the top, is to hold 500 cm^3 of beer. Find the height and the radius that minimize the amount of material needed to manufacture the can.

5. An open box is to be made from a 3 cm by 8 cm rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sizes. Find the maximum volume that the open box can have.

6. If the position S of a particle moving along an s-axis is given as a function of the time t by $S(t) = 2t^3 - 9t^2 + 12t$ for $t > 0$,

   (a) find the velocity, $v(t)$ and acceleration, $a(t)$ of the particle;

   (b) find the average velocity, v_{av} of the particle over the time interval $t_1 = 1$ and $t_2 = 2$.

   (c) find all time intervals when the particle moves in the positive direction and when it moves in the negative direction. When is it stopped?

   (d) find all time intervals for $t > 0$ when the particle is speeding up and when it is slowing down.

Solutions

1. Graph of $f(x) = \frac{x^2-9}{x^2-4}$.

2. The function $f(x) = (x^2+x)^{\frac{2}{3}}$ is everywhere continuous. It is in particular continuous on the closed, finite interval $[-2, 3]$. Therefore, $f(x)$ has both an absolute maximum and an absolute minimum on $[-2, 3]$.

   (a)
   $$f'(x) = \left((x^2+x)^{\frac{2}{3}}\right)' = \frac{2}{3}(x^2+x)'(x^2+x)^{-\frac{1}{3}} = \frac{2(2x+1}{3(x^2+x)^{\frac{1}{3}}}.$$
   $f'(x)$ is undefined at $x = 0$ and $x = -1$, i.e., 0 and -1 are singular points of $f(x)$. $f'(x) = 0 \Longrightarrow 2x+1 = 0 \Longrightarrow x = -\frac{1}{2}$, i.e., $-\frac{1}{2}$ is a critical point.

   (b) The singular points 0 and -1, the critical point $-\frac{1}{2}$ belong all to the interval $[-2, 3]$ which has endpoints -2 and 3. We have:
   $f(0) = 0 = f(-1)$, $f(-\frac{1}{2}) = (\frac{1}{4})^{\frac{2}{3}}$, $f(-2) = 2^{\frac{2}{3}}$ and $f(3) = 12^{\frac{2}{3}}$. Among these values of f, 0 is the smallest and $12^{\frac{2}{3}}$ is the largest, i.e., 0 is the absolute minimum value of f occurring at the points $x = 0$ and $x = -1$, and $12^{\frac{2}{3}}$ is the absolute maximum value of f on $[-2, 3]$ attained at 3.

3. (a) The function $f(x) = x^3 - 3x^2 - 9x$ is everywhere differentiable; so, it has no singular points. $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3) = 0 \Longrightarrow x = -1 \quad \text{or} \quad x = 3$, i.e., $f(x)$ has two critical points $x = -1$ and $x = 3$. Moreover, there is a sign change for $f'(x)$ at -1 from + (left) to - (right) and at 3 from - (left) to + (right), i.e.,

$f(x)$ has a relative (local) maximum at -1 ($(-1, 5)$ is a rel. max point) and a relative minimum at 3 ($(3, -27)$ is a rel. min. point).

(b) $f''(x) = 6x - 6 = 6(x-1) = 0 \Longrightarrow x = 1$. $f''(x) < 0$ on $(-\infty, 1)$ and $f''(x) > 0$ on $(1, \infty)$, i.e, f is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$.

(c) Since $f''(x)$ changes its concavity at 1, $f(x)$ has an inflection at 1, i.e., $(1, -11)$ is an inflection point to the graph of $f(x)$.

4. Denote by h and r the height and radius of the open can, respectively. Then the volume of the can is $V = \pi r^2 h = 500$. This implies $h = \frac{500}{\pi r^2}$. The total area is $S = 2\pi h r^2 + \pi r^2 = \pi r^2 + \frac{1000}{r}$. The problem is to minimize $S = S(r)$ (i.e., find the absolute minimum value for V) on $(0, +\infty)$. Since $\lim_{r\to 0^+} S(r) = +\infty = \lim_{r\to\infty} S(r)$, $S(r)$ has an absolute minimum on $(0, +\infty)$, occurring at a critical point as $S(r)$ is differentiable on $(0, +\infty)$, i.e., has no singular point.

$$S'(r) = 2\pi r - \frac{1000}{r^2} = \frac{2\pi r^3 - 1000}{r^2}.$$

$S'(r) = 0 \Longrightarrow r = r_{min} = \frac{10}{\sqrt[3]{2\pi}}$ is the only critical point. It is the point where (r) takes its absolute minimum. And the minimum height is $h = h_{min} = \frac{500}{\pi r_{min}^2} = \frac{500}{\pi(\frac{10}{\sqrt[3]{2\pi}})^2} = \frac{10}{\sqrt[3]{2\pi}} = r_{min}$. Therefore, to minimize the cost of the material needed to make the can, the height and the radius must have the same length.

5. If we cut an x by x square from each corner, then the resulting box will be $8-2x$ long, $3-2x$ wide and x high. Its

volume is $V = V(x) = x(8-2x)(3-2x)$. The problem is to maximize the volume V on $[0, \frac{3}{2}]$. $V(x)$ is differentiable everywhere, it has no singular point. $V'(x) = 12x^2 - 44x + 24 = 4(3x-2)(x-3)$. So, $V(x)$ has one critical point $\frac{2}{3} \in [0, \frac{3}{2}]$. $V(0) = 0 = V(\frac{3}{2})$ and $V(\frac{2}{3}) = \frac{200}{27}$. Thus the open box can have a maximum volume of $\frac{200}{27}\ cm^3$.

6. The position function is $S(t) = 2t^3 - 9t^2 + 12t$ for $t > 0$.

   (a) $v(t) = \frac{d}{dt}S(t) = (2t^3 - 9t^2 + 12t)' = 6(t-1)(t-2)$ is the velocity. The acceleration is $v(t) = \frac{d}{dt}v(t) = 12t - 18 = 6(2t-3)$.

   (b) The average velocity from $t_1 = 1$ to $t_2 = 2$ is

$$\begin{aligned} a_{av} &= \frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{S(2) - S(1)}{2-1} \\ &= 2(2)^3 - 9(2)^2 + 12(2) - (2 - 9 + 12) \\ &= -1 \text{ Unit.} \end{aligned}$$

   (c) The particle moves in positive direction when $v(t) > 0$, i.e., during the time interval $(0, 1)$ and $(2, +\infty)$, and in the negative direction when $v(t) < 0$, i.e., on $(1, 2)$. It is momentarily stopped at time t when $v(t) = 0$, i.e., at $t = 1$ and $t = 2$.

   (d) $a(t)$ and $v(t)$ change sign at 1, $\frac{3}{2}$, and 2. During the time intervals $(0, 1)$ and $(\frac{3}{2}, 2)$, $a(t)$ and $v(t)$ have opposite sign, the particle slows down on these time intervals. It speeds up when $a(t)$ and $v(t)$ have the same sign, i.e., on $(1, \frac{3}{2})$ and $(2, \infty)$.