THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 249-01

Fall, 2008 Time: 2 Hours
December 9, 2008

1. Solve for x :

   (a) $\frac{1}{2}\ln(3x+1)-\ln(2x)=0 \qquad \ln\sqrt{(3x+1)}-\ln(2x)=0 \qquad \ln\frac{\sqrt{3x+1}}{2x}=0$

   apply exp.function $\quad \frac{\sqrt{3x+1}}{2x}=e^0=1 \qquad \sqrt{3x+1}=2x$ square

   $3x+1=4x^2 \qquad 4x^2-3x-1=(4x+1)(x-1)=0$

   possible solution $x=1$ and $x=-\frac{1}{4}$ but log is defined only for positive numbers

   so the only solution is $x=1$

   (b) $2^{2x-1}=\frac{5}{3^x}$ apply ln function: $\quad \ln 2^{2x-1}=\ln\frac{5}{3^x}=\ln 5-\ln 3^x$

   $(2x-1)\ln 2=\ln 5-x\ln 3 \qquad 2x\ln 2-\ln 2=\ln 5-x\ln 3$

   $x(2\ln 2+\ln 3)=\ln 5+\ln 2 \qquad x\ln\left(2^2\cdot 3\right)=\ln 5\cdot 2\rightarrow x=\frac{\ln 10}{\ln 12}.$

2. Find the domain of definition and the derivative of f if

   (a) for $x\neq-\frac{1}{2}$ by Quotient Rule

   $f'(x)=\left(\frac{\cos(2x)}{2x+1}\right)'=\frac{-2\sin 2x\,(2x+1)-2\cos 2x}{(2x+1)^2};$

   (b) for $1-2x>0 \qquad x<\frac{1}{2}$ by Product Rule

   $f'(x)=\left[e^{x^2}\cdot\ln(1-2x)\right]'=e^{x^2}\left(x^2\right)'\cdot\ln(1-2x)+e^{x^2}\cdot\frac{(1-2x)'}{1-2x}$

   $=2xe^{x^2}\ln(1-2x)+e^{x^2}\cdot\frac{-2}{1-2x};$

   (c) for $x>0$ by Chain Rule

   $f'(x)=\left(5^{\sqrt{x}}\right)'=5^{\sqrt{x}}\ln 5\cdot\left(\sqrt{x}\right)'=5^{\sqrt{x}}\cdot\frac{\ln 5}{2\sqrt{x}}.$

3. (a) Find the tangent approximation (linearization) of $f(x)=\frac{1}{\sqrt[3]{x}}$ around $x_0=27$.

   $f(27)=\frac{1}{\sqrt[3]{27}}=\frac{1}{3} \qquad f'(x)=\left(x^{-\frac{1}{3}}\right)'=-\frac{1}{3}x^{-\frac{4}{3}}$

   $f'(27)=\frac{-1}{3\left(\sqrt[3]{27}\right)^4}=\frac{-1}{3^5}=\frac{-1}{243}$ thus $\qquad L(x)=\frac{1}{3}-\frac{1}{243}(x-27)$

(b) Use it to estimate $\frac{1}{\sqrt[3]{25}}$: $f(x) \doteq L(x)$ for x close to 27

for $x = 25$ $\qquad f(25) \doteq L(25)$

$\frac{1}{\sqrt[3]{25}} \doteq L(25) = \frac{1}{3} + \frac{2}{243} = \frac{83}{243} = 0.341564$

4. For $y = \frac{x}{(x+3)^2}$

(a) the domain,:$\{x \neq -3\}$, $x = -3$ is V.A.since $\lim\limits_{x\to -3} \frac{x}{(x+3)^2} = ”\frac{-3}{0^+}” = -\infty$

for horizontal asympt. $\lim\limits_{x\to\pm\infty} \frac{x}{(x+3)^2} = ”\frac{\pm\infty}{+\infty}” L'H. \lim\limits_{x\to\pm\infty} \frac{1}{2(x+3)} = ”\frac{1}{\pm\infty}” = 0$

$y = 0$ H.A. both ends; $\qquad x = 0 \rightleftarrows y = 0$ intercept

(b) $y' = \frac{3-x}{(x+3)^3} \to x = -3$ sing point $\qquad x = 3$ critical point

testing : $y' - -^{neg} - -_{-3} -^{pos} - - -_{3} -^{neg} - - -$

so f is increasing on $(-3, 3)$and decr. on $(-\infty, -3)$ and on $(3, +\infty)$

if $\quad x = 3 \qquad y = \frac{1}{12} \qquad$ local maximum

(c) $y'' = \frac{2x-12}{(x+3)^4} \to x = -3$ sing point $\qquad x = 6$ possible inflection point

testing : $y'' - -^{neg} - -_{-3} -^{neg} - - -_{6} -^{pos} - - -$

f is concave down on $(-\infty, -3)$ and on $(-3, 6)$ and concave up on$(6, +\infty)$

(d) No local minimum and absolute maximum at $x = 3$, the range$\left(-\infty, \frac{1}{12}\right]$

5. Find the dimensions of the most economical cylindrical can holding the volume $V = 4\pi$ cm^3 if the material for the lid and bottom costs 4 cents per cm^2 and for the side 2 cents per cm^2.

variables: r...radius $\qquad h$...height

given volume $V = \pi r^2 h = 4\pi \to h = \frac{4}{r^2}$

lid and bottom circles so each area is $\qquad \pi r^2$

the side is a rectangle with length $2\pi r$ and height h so area is $(2\pi rh)$

cost=4 (area of lid and bottom) + 2 (area of the side) $= 4(\pi r^2 \cdot 2) + 2(2\pi rh) =$

$= 8\pi r^2 + 4\pi rh$ $\qquad$ reduce to one variable by substituting for h

$f(r) = 8\pi r^2 + 4\pi r \cdot \frac{4}{r^2} = 8\pi\left(r^2 + \frac{2}{r}\right), r > 0$

for critical points : $f'(r) = 8\pi\left(2r - \frac{2}{r^2}\right) = 16\pi\frac{r^3-1}{r^2} = 0$ for $r = 1$

to justify minimum $f''(r) = 8\pi\left(2 + \frac{4}{r^3}\right) > 0$ for $r > 0$ so f is conc.up

together $\qquad r = 1cm \qquad h = 4cm.$

6. graph

7. Find

(a) for $x > 0$ using $(A-B)^2 = A^2 - 2AB + B^2$

$$\int (\sqrt{x} - 2x)^2\, dx = \int (\sqrt{x})^2\, dx - \int 2\sqrt{x} \cdot 2x dx + \int (2x)^2\, dx =$$

$$= \int x dx - 4 \int x^{\frac{3}{2}} dx + 4 \int x^2 dx = \frac{x^2}{2} - \frac{8}{5} x^{\frac{5}{2}} + \frac{4x^3}{3} + c$$

(b) for $x \neq -\frac{1}{3}$ by subst. $3x + 1 = u$ $\quad 3dx = du \quad dx = \frac{1}{3} du$

$$\int \frac{1}{3x+1} dx = \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln|u| + c = \frac{1}{3} \ln|3x+1| + c$$

8. Evaluate

(a) by subst $u = \sin x$ $\quad du = \cos x dx \quad x = \frac{\pi}{4} \to u = \frac{1}{\sqrt{2}}; x = 0 \to u = 0$

$$\int_0^{\frac{\pi}{4}} \cos x \cdot \sin^3 x\; dx = \int_0^{\frac{1}{\sqrt{2}}} u^3 dy = \left[\frac{u^4}{4}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{4}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 0\right] = \frac{1}{16}$$

also possible by $u = \cos x \quad du = -\sin x dx \quad \sin^2 x = 1 - u^2$

(b) by subst. $u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}} dx \quad 2du = \frac{dx}{\sqrt{x}}$

$x = 0 \to u = 0 \quad x = 4 \to u = 2$

$$\int_0^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\; dx = 2\int_0^2 2^u du = \frac{2}{\ln 2}\left[2^u\right]_0^2 = \frac{2}{\ln 2}\left[2^2 - 1\right] = \frac{6}{\ln 2}.$$

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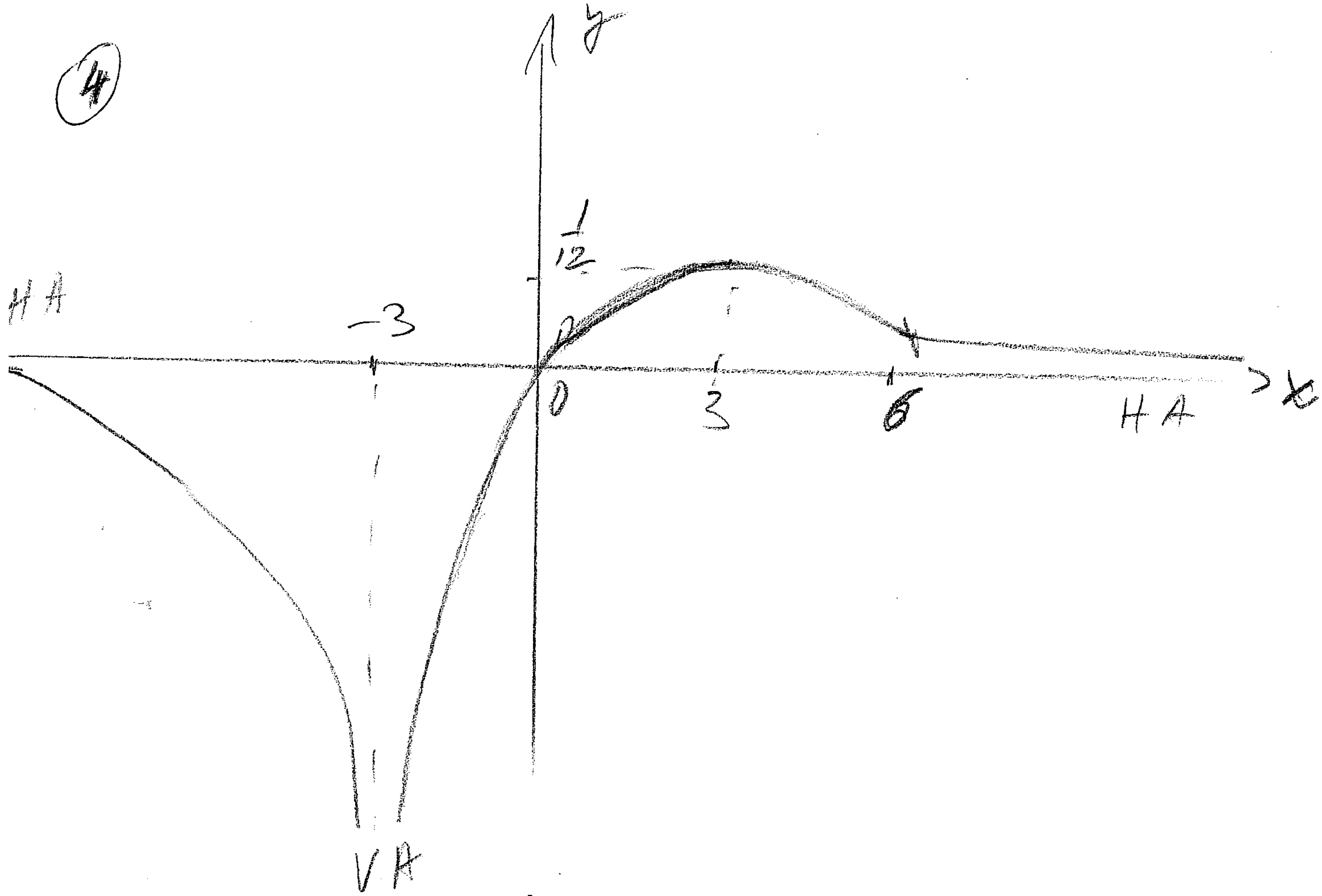

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