

MATH 251/249
Midterm Handout

For 1)

a) $\lim_{x \rightarrow \infty} (x^2 - x^2 \cos \frac{1}{x}) = \lim_{x \rightarrow \infty} x^2 (1 - \cos \frac{1}{x}) \text{ "}\infty \cdot 0\text{"}$ $u = \frac{1}{x}, u \rightarrow 0^+$

so

$$\begin{aligned} \lim_{x \rightarrow \infty} \dots &= \lim_{u \rightarrow 0^+} \frac{(1 - \cos u)}{u^2} \cdot \frac{(1 + \cos u)}{(1 + \cos u)} = \lim_{u \rightarrow 0^+} \frac{(1 - \cos^2 u)}{u^2 (1 + \cos u)} = \\ &= \lim_{u \rightarrow 0^+} \frac{\sin^2 u}{u^2 (1 + \cos u)} = \lim_{u \rightarrow 0^+} \left(\frac{\sin u}{u} \right)^2 \cdot \lim_{u \rightarrow 0^+} \frac{1}{(1 + \cos u)} = 1 \cdot \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

b) $\lim_{x \rightarrow 0} (x^2 - x^2 \cos \frac{1}{x}) = \lim_{x \rightarrow 0} x^2 (1 - \cos \frac{1}{x}) \text{ "}0 \cdot DNE\text{"} = 0$ by Sq,Th

since

$$0 \leq 1 - \cos \theta \leq 2 \text{ for any } \theta \quad 0 \leq x^2 (1 - \cos \frac{1}{x}) \leq 2x^2 \rightarrow 0 \text{ as } x \rightarrow 0.$$

For 2)

a) $\lim_{x \rightarrow 0} \frac{\sin x}{x - \pi} = \frac{0}{-\pi} = 0$

b) $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi} = \text{"}0\text{"}$, $= \lim_{u \rightarrow 0} \frac{\sin(u + \pi)}{u} = \lim_{u \rightarrow 0} \frac{\sin u \cos \pi + \cos u \sin \pi}{u} = \lim_{u \rightarrow 0} -\frac{\sin u}{u} = -1$

where $x - \pi = u, x = u + \pi$ and $\cos \pi = -1, \sin \pi = 0$

c) $\lim_{x \rightarrow -\infty} \frac{\sin x}{x - \pi} = 0$ by Squeeze Th. since $-1 \leq \sin x \leq 1$ and $x - \pi < 0$

$$\frac{-1}{x - \pi} \geq \frac{\sin x}{x - \pi} \geq \frac{1}{x - \pi} \text{ and } \lim_{x \rightarrow -\infty} \frac{\pm 1}{x - \pi} = 0.$$

For 3)

if $x = \frac{-1}{2}$ then $y = \frac{\cos(\frac{-\pi}{2})}{\frac{3}{2}} = 0$ so $y = m(x + \frac{1}{2})$

for m use Q.R. to find $y' = \left(\frac{\cos \pi x}{1 - x} \right)' = \frac{-\pi \sin \pi x \cdot (1 - x) - \cos \pi x \cdot (-1)}{(1 - x)^2}$

at $x = -\frac{1}{2}$ $\cos \frac{-\pi}{2} = 0, \sin \frac{-\pi}{2} = -1$ so

$$m = \frac{-\pi(-1) \cdot \frac{3}{2} - 0}{\left(\frac{3}{2}\right)^2} = \frac{\pi}{\frac{3}{2}} = \frac{2\pi}{3} \dots \text{slope} \quad \text{tangent } y = \frac{2\pi}{3} \left(x + \frac{1}{2}\right).$$

For 4)

for $y = \left(\sin \frac{1}{\sqrt{x^4+1}}\right)^3$ use Chain Rule twice

$$\begin{aligned} y' &= 3 \left(\sin \frac{1}{\sqrt{x^4+1}}\right)^2 \left(\sin \frac{1}{\sqrt{x^4+1}}\right)' = 3 \left(\sin \frac{1}{\sqrt{x^4+1}}\right)^2 \left(\cos \frac{1}{\sqrt{x^4+1}}\right) \left((x^4+1)^{-\frac{1}{2}}\right)' = \\ &= 3 \left(\sin \frac{1}{\sqrt{x^4+1}}\right)^2 \left(\cos \frac{1}{\sqrt{x^4+1}}\right) \left(-\frac{1}{2}\right) \left((x^4+1)^{-\frac{3}{2}}\right) 4x^3 = \frac{-6x^3 \sin^2 \frac{1}{\sqrt{x^4+1}} \cos \frac{1}{\sqrt{x^4+1}}}{(x^4+1)^{\frac{3}{2}}} \end{aligned}$$

For 5)

the function $f(x) = x - 2 \sin(\pi x)$ is continuous everywhere and

$$f\left(\frac{1}{2}\right) = \frac{1}{2} - 2 \sin \frac{\pi}{2} = -\frac{3}{2} < 0 \text{ and } f(1) = 1 - 2 \sin \pi = 1 > 0$$

so by IVT there must be an c between $\frac{1}{2}$ and 1 where $f(c) = 0$

For 6)

the polynomial $p = 2x^3 - 6x^2 + 7$ is continuous everywhere and $p(0) = 7 > 0$

and $p(-1) = -2 - 6 + 7 = -1 < 0$ so by IVT(intermediate value theorem

there must be one root r_1 between -1 and 0 $r_1 \in (-1, 0)$

since $p(1) = 2 - 6 + 7 = 3 > 0$ and $p(2) = 16 - 24 + 7 = -1 < 0$

so by IVT there is another root between 1 and 2 $r_2 \in (1, 2)$

finally $p(3) = 54 - 54 + 7 = 7 > 0$

by IVT there is a root $r_3 \in (2, 3)$

For 7)

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta} = \pm \sqrt{1 - \frac{1}{25}} = \pm \sqrt{\frac{24}{25}} = \pm \frac{\sqrt{24}}{5}$$

but since θ is in the second quadrant \cos must be negative

$$\sec \theta = \frac{1}{\cos \theta} = -\frac{5}{\sqrt{24}}$$

For 8)

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = \pm \sqrt{1 - \frac{4}{9}} = \pm \sqrt{\frac{5}{9}} = \pm \frac{\sqrt{5}}{3}$$

but since θ is in the fourth quadrant \sin must be negative

$$\sin \theta = -\frac{\sqrt{5}}{3}. \text{ Now } \sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \left(-\frac{\sqrt{5}}{3}\right) \cdot \frac{2}{3} = -\frac{4\sqrt{5}}{9}.$$

For 9)

$$f(x) = \begin{cases} \left(\frac{2}{2x+1} - 3\right)(4x^2 - 1) & \text{for } x < -\frac{1}{2} \\ ax + b & \text{for } -\frac{1}{2} \leq x \leq 2 \\ \cos\left(-\frac{\pi}{x}\right) & \text{for } x > 2 \end{cases}.$$

the function is continuous except $x = -\frac{1}{2}$ and $x = 2$

$$f\left(-\frac{1}{2}\right) = \frac{-1}{2}a + b = \lim_{x \rightarrow -\frac{1}{2}^+} f(x)$$

$$\begin{aligned} \lim_{x \rightarrow -\frac{1}{2}^-} f(x) &= \lim_{x \rightarrow -\frac{1}{2}^-} \left(\frac{2}{2x+1} - 3\right)(4x^2 - 1) = \lim_{x \rightarrow -\frac{1}{2}^-} \left(\frac{2-3(2x+1)}{2x+1}\right)(4x^2 - 1) \\ &= \lim_{x \rightarrow -\frac{1}{2}^-} \frac{2-6x-3}{2x+1} \cdot (2x-1)(2x+1) = \lim_{x \rightarrow -\frac{1}{2}^-} (-1-6x)(2x-1) = 2 \cdot (-2) = -4 \end{aligned}$$

All 3 numbers must be the same so $\frac{-1}{2}a + b = -4$

Similarly, for $x = 2$

$$f(2) = 2a + b = \lim_{x \rightarrow 2^-} f(x) \text{ and } \lim_{x \rightarrow 2^+} \cos \frac{-\pi}{x} = \cos \frac{-\pi}{2} = 0$$

All 3 numbers must be the same so $2a + b = 0 \rightarrow b = -2a$

back to $\frac{-1}{2}a + b = -4$ and $a\left(\frac{-1}{2} - 2\right) = -4 \Rightarrow a\left(\frac{-5}{2}\right) = -4$

$$\text{so } a = (-4) \left(\frac{-2}{5}\right) = \frac{8}{5} \text{ and } b = -2a = \frac{-16}{5}.$$

For 10)

$$f(x) = \begin{cases} \cos(\pi x) - 2 \sin \frac{\pi x}{2} & \text{for } x > 3 \\ ax^2 + b & \text{for } 0 \leq x \leq 3 \\ 6 \cdot \frac{\sqrt{9-x} - 3}{x} & \text{for } x < 0 \end{cases}$$

the function is continuous for any x except $x = 3$ and $x = 0$

$$f(3) = 9a + b = \lim_{x \rightarrow 3^-} f(x) \text{ and}$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \left[\cos(\pi x) - 2 \sin \frac{\pi x}{2}\right] = \cos(3\pi) - 2 \sin \frac{3\pi}{2} = -1 - 2(-1) = 1$$

All 3 numbers must be the same so $9a + b = 1$

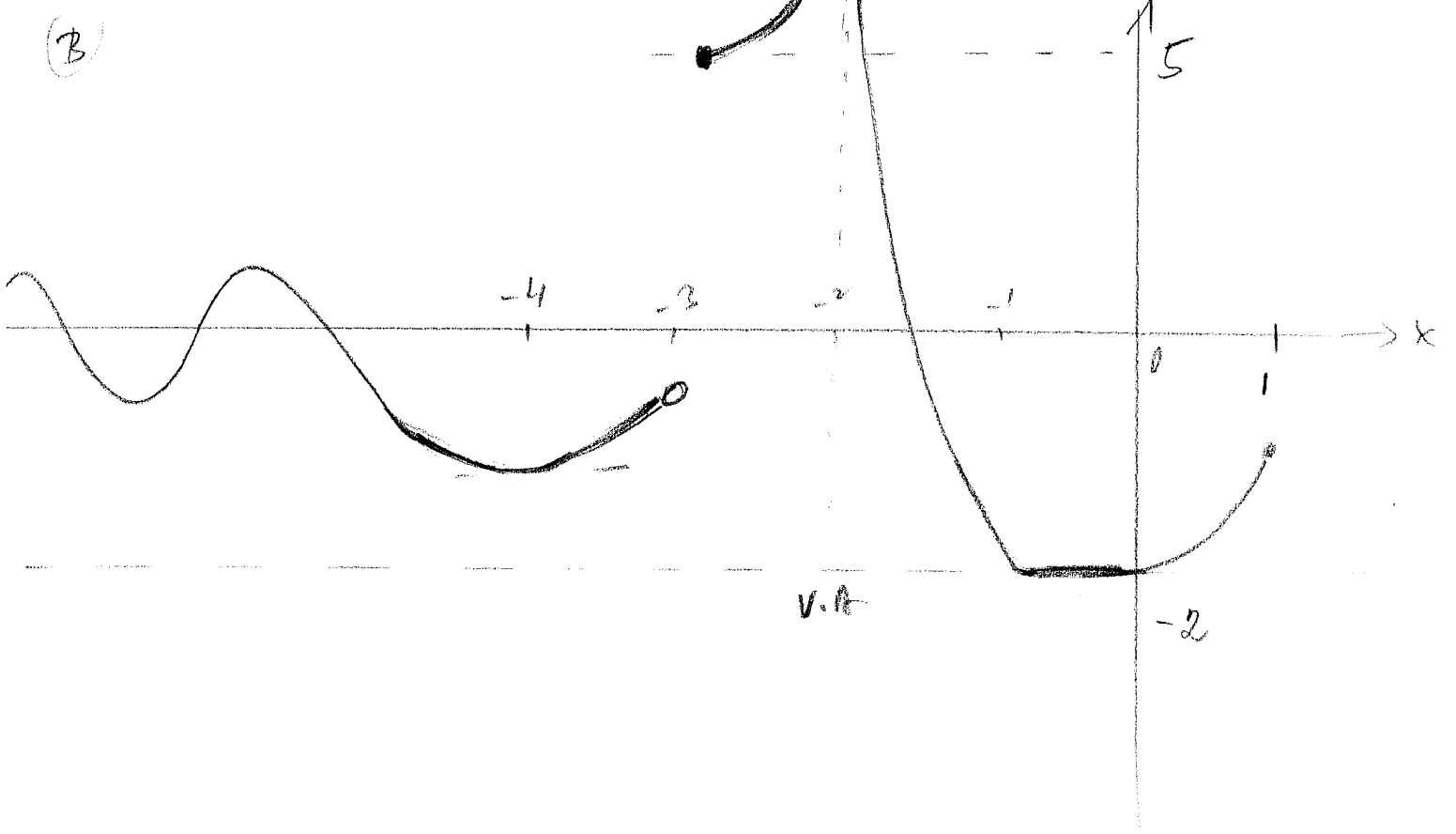
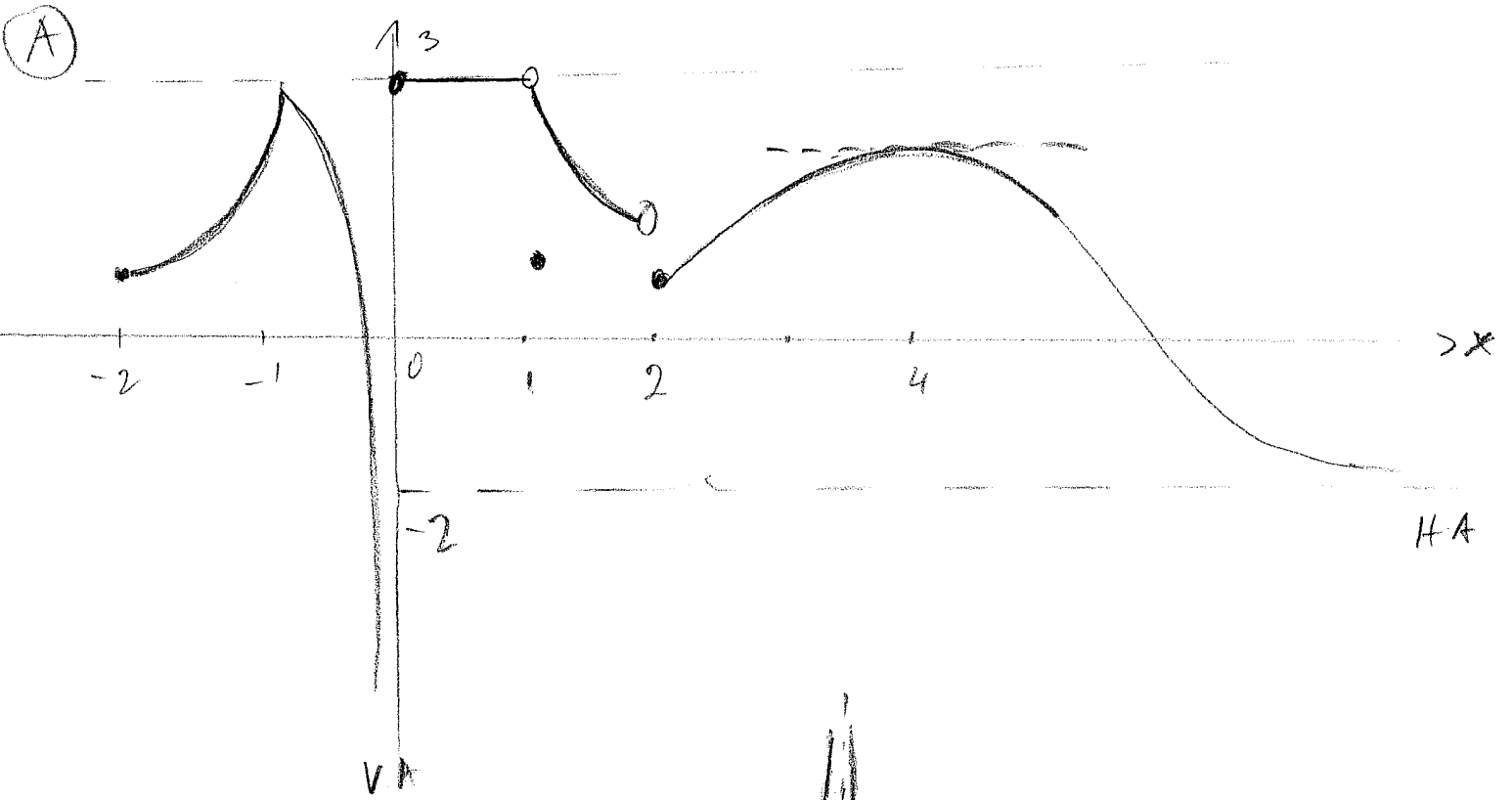
Similarly, for $x = 0$

$$\begin{aligned} f(0) = b &= \lim_{x \rightarrow 0^+} f(x) \quad \text{and} \quad \lim_{x \rightarrow 0^-} 6 \cdot \frac{\sqrt{9-x}-3}{x} \cdot \frac{\sqrt{9-x}+3}{\sqrt{9-x}+3} = 6 \lim_{x \rightarrow 0^-} \frac{9-x-3^2}{x(\sqrt{9-x}+3)} = \\ &= 6 \lim_{x \rightarrow 0^-} \frac{-x}{x(\sqrt{9-x}+3)} = 6 \lim_{x \rightarrow 0^-} \frac{-1}{(\sqrt{9-x}+3)} = \frac{-6}{6} = -1 \end{aligned}$$

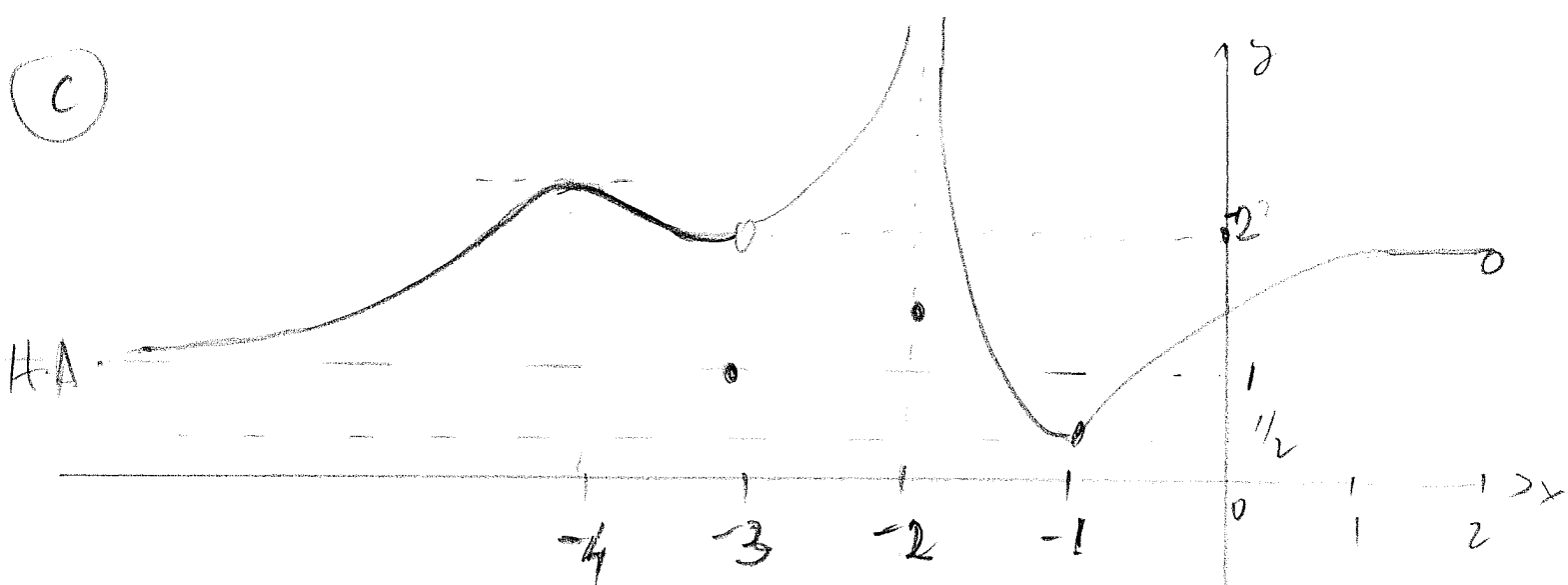
All 3 numbers must be the same so $b = -1$

substitute $b = -1$ in the first equation $9a - 1 = 1$ $a = \frac{2}{9}$ and $b = -1$.

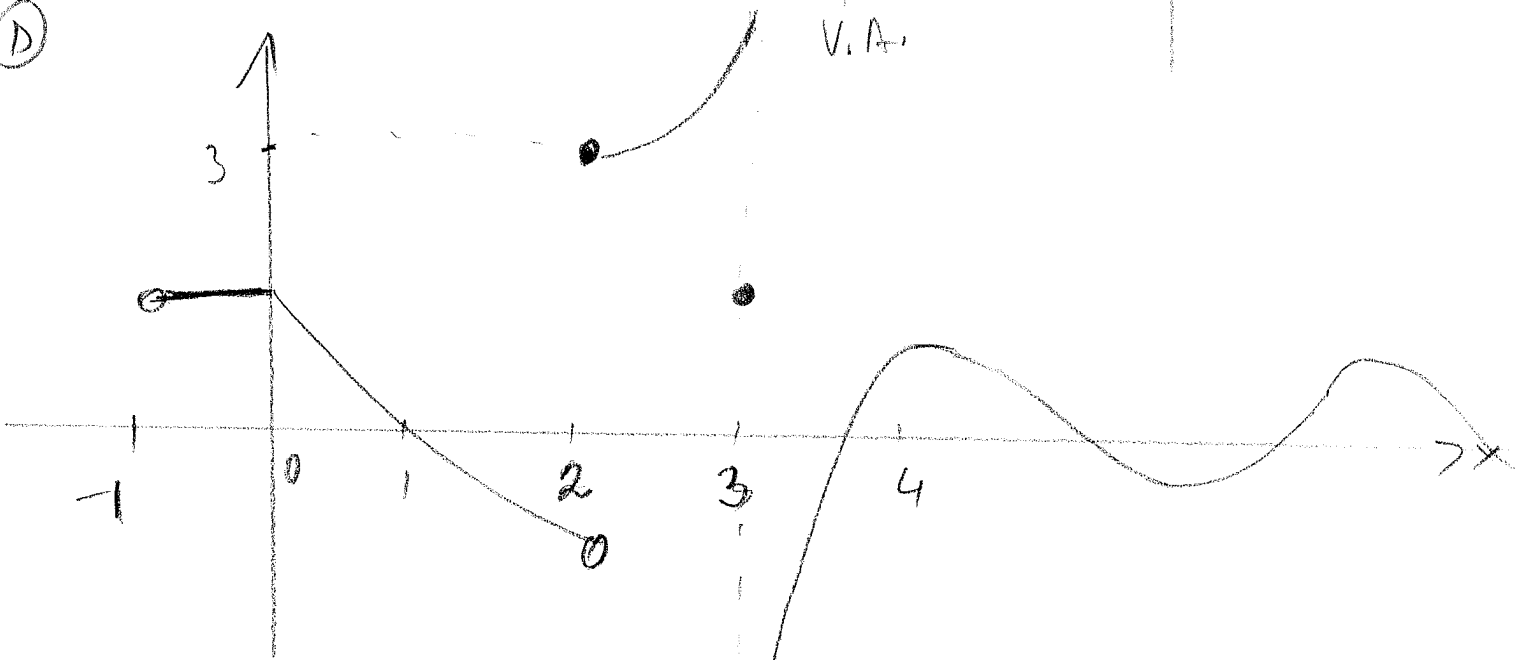
For 11)



(C)



(D)



(E)

