The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #2

Solution.

For 1a) For $f(x)=\dfrac{1}{1-x}\left(1-\dfrac{4}{x+3}\right)$
the type of the limit is $"\frac{0}{0}"$as $x\to 1$ so we have to simplify
$f(x)=\dfrac{1}{1-x}\cdot\dfrac{x+3-4}{x+3}=\dfrac{-(1-x)}{(1-x)(x+3)}=\dfrac{-1}{x+3}$ for any $x\neq 1,-3$.
As $x\to 1$ the limit is $\frac{-1}{4}$.
For1b
as $x\to -3^{+}$ $\quad x+3>0$
we can use the simplification from above and the type of the limit is $"\frac{-1}{0^{+}}"$
so the limit is $-\infty$. OR
from the original formula the limit is $L=\frac{1}{4}\cdot\left(1-"\frac{4}{0^{+}}"\right)=-\infty$
For 1c)
as $x\to +\infty$.
From the simplified formula the type is $"\frac{-1}{\infty}"$ so the limit is 0.
OR
from the original the type is $"\frac{1}{-\infty}"\cdot\left(1-"\frac{4}{\infty}"\right)=0\cdot(1-0)=0$.
For 2)
For $f(x)=\sqrt{9-x^{2}}$ and $g(x)=\dfrac{3}{x-1}$
first the domains of the given functions for D_f solve $9-x^{2}\geq 0,(3-x)(3+x)\geq 0$
parabola open down, above the x-axis between roots OR
split points are $x=\pm 3$, testing $--^{neg}--_{-3}---^{pos}---_{3}--^{neg}---$
so $D_f=[-3,3]$ $\qquad D_g=\{x\neq 1\}$ since $x-1\neq 0$.
$g\circ g(x)=g(g(x))=\dfrac{3}{(...)-1}=\dfrac{3}{\frac{3}{x-1}-1}=\dfrac{3}{\frac{3-(x-1)}{x-1}}=\dfrac{3(x-1)}{4-x}$
we must start in D_g i.e. $x\neq 1$ and we have to guarantee that
$4-x\neq 0$ so $x\neq 4$ $D_{g\circ g}=\{x\neq 1\wedge x\neq 4\}$
$f\circ g(x)=\sqrt{9-(..)^{2}}=\sqrt{9-\frac{9}{(x-1)^{2}}}=\sqrt{9\cdot\left(1-\frac{1}{(x-1)^{2}}\right)}=3\cdot\sqrt{\frac{x^{2}-2x+1-1}{(x-1)^{2}}}=3\sqrt{\frac{x(x-2)}{(x-1)^{2}}}$
we must start in $\mathrm{D}_g, x\neq 1$ and quarantee that$\dfrac{x(x-2)}{(x-1)^{2}}\geq 0$,split points are $x=0,2,1$
testing $--^{pos}-_{0}--^{negl}--_{1}--^{neg}--_{2}-^{pos}---$so $\qquad D_{f\circ g}=(-\infty,0]\cup[2,\infty)$.
For 3)
For $g(x)=\dfrac{4}{2x-8}$ and $f(x)=\sqrt{x^{2}-9}$
$D_g=\{x\neq 4\}, D_f=(-\infty,-3]\cup[3,+\infty)$ since we have to solve: $x^{2}-9\geq 0$
$(x-3)(x+3)\geq 0$ $\qquad$ parabola open up with roots $x=\pm 3$
$\quad OR \quad x^{2}\geq 9 \quad \sqrt{x^{2}}=|x|\geq 3$
Now, $g\circ g(x)=\dfrac{4}{2()-8}=\dfrac{2\cdot 2}{2\left[\left(\frac{4}{2x-8}\right)-4\right]}=\dfrac{2}{\frac{4-8x+32}{2x-8}}=2\cdot\dfrac{2x-8}{36-8x}=\dfrac{4(x-4)}{4(9-2x)}=\dfrac{x-4}{9-2x}$

for $x \neq 4$ and $x \neq \frac{9}{2}$,so $D_{g \circ g} = (-\infty, 4) \cup (4, 4.5) \cup (4.5, +\infty)$.

For $g \circ f(x) = \dfrac{4}{2\,() - 8} = \dfrac{4}{2\sqrt{x^2 - 9} - 8}$ = you can simplify

$$= \frac{2}{\sqrt{x^2-9}-4} \cdot \frac{\sqrt{x^2-9}+4}{\sqrt{x^2-9}+4} = \frac{2\left(\sqrt{x^2-9}+4\right)}{x^2-9-4^2} = \frac{2\left(\sqrt{x^2-9}+4\right)}{x^2-25}$$

we know that $x \in D_f$ and that new denominator must be non-zero

$\sqrt{x^2-9} - 4 \neq 0, \sqrt{x^2-9} \neq 4$, so $x^2 - 9 \neq 4^2, x^2 \neq 25$

OR after simplification $x^2 - 25 \neq 0$ i.e. $x \neq \pm 5$,together

$D_{g \circ f} = (-\infty, -5) \cup (-5, -3] \cup [3, 5) \cup (5, +\infty)$. .

For 4a)

for $f(x) = \dfrac{1 - 4x^2}{6x^2 - 5x + 1}$ as $x \to -\infty$ the type is "$\frac{-\infty}{\infty}$" so

divide top and bottom by x^2 :

$$\lim_{x \to -\infty} \frac{\frac{1}{x^2} - 4}{6 - \frac{5}{x} + \frac{1}{x^2}} = \frac{0 - 4}{6 - 0 + 0} = -\frac{4}{6} = -\frac{2}{3} \text{ (since "}\tfrac{1}{\pm\infty}\text{"} = 0)$$

For 4b)

as $x \to \frac{1}{2}$ the type is "$\frac{0}{0}$" and we have polynomials so factorize

$$\lim_{x \to \frac{1}{2}} \frac{(1 - 2x)(1 + 2x)}{(2x - 1)(3x - 1)} = \lim_{x \to \frac{1}{2}} \frac{-(1 + 2x)}{3x - 1} = \frac{-2}{\frac{1}{2}} = -4.$$

For 4c) as $x \to \frac{1}{3}^-$

we can use the simplification from above but the type is "$\frac{neg\#}{0^-}$" since $x < \frac{1}{3}$ so $3x - 1 < 0$

$$\lim_{x \to \frac{1}{3}^-} \frac{-(1 + 2x)}{3x - 1} = \text{"}\frac{-\frac{5}{3}}{0^-}\text{"} = \text{"}\frac{1}{0^+}\text{"} = +\infty$$

For 5a)

For $\dfrac{\sqrt{3x} - 3}{\sqrt{2x^2 - 6x}}$ as $x \to 3^+$

$$\lim_{x \to 3^+} \frac{\sqrt{3x} - 3}{\sqrt{2x^2 - 6x}} = \text{"}\frac{0}{0}\text{"} = \lim_{x \to 3^+} \frac{\sqrt{3x} - 3}{\sqrt{2x^2 - 6x}} \cdot \frac{\sqrt{3x} + 3}{\sqrt{3x} + 3} = \lim_{x \to 3^+} \frac{3x - 3^2}{\sqrt{2x^2 - 6x}} \cdot \frac{1}{\sqrt{3x} + 3} =$$

$$= \lim_{x \to 3^+} \frac{3(x - 3)}{\sqrt{2x(x - 3)}} \cdot \frac{1}{\sqrt{3x} + 3} = \lim_{x \to 3^+} \frac{3}{\sqrt{2x}} \cdot \frac{x - 3}{\sqrt{x - 3}} \cdot \frac{1}{\sqrt{3x} + 3} =$$

$$= \lim_{x \to 3^+} \frac{3}{\sqrt{2x}} \cdot \sqrt{x - 3} \cdot \frac{1}{\sqrt{3x} + 3} = \frac{3}{\sqrt{6}} \cdot 0 \cdot \frac{1}{6} = 0.$$

For 5b) as $x \to +\infty$,

the type is "$\frac{\infty}{\infty}$" so we have to divide by the highest power in the denominator

in the original from by $x = \sqrt{x^2}$ $(x > 0)$:

$$\lim_{x \to +\infty} \frac{\sqrt{3x} - 3}{\sqrt{2x^2 - 6x}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} = \lim_{x \to +\infty} \frac{\sqrt{\frac{3}{x}} - \frac{3}{x}}{\sqrt{2 - \frac{6}{x}}} = \frac{0 - 0}{\sqrt{2}} = 0$$

OR we can used the simplified expresion from a)

$$\lim_{x \to +\infty} \frac{3}{\sqrt{2x}} \cdot \sqrt{x - 3} \cdot \frac{1}{\sqrt{3x} + 3} = \lim_{x \to +\infty} 3\sqrt{\frac{x-3}{2x}} \cdot \frac{1}{\sqrt{3x}+3} =$$

$$= \lim_{x \to +\infty} 3\sqrt{\frac{1}{2} - \frac{3}{2x}} \cdot \lim_{x \to +\infty} \frac{1}{\sqrt{3x}+3} = \frac{3}{\sqrt{2}} \cdot 0 = 0 \text{ since "}\tfrac{1}{\infty}\text{"} = 0.$$

For 5c) as $x \to 0$

the limit DNE (does not exist neither as a number nor as $\pm\infty$)

since the function is not defined for small negative x $(\sqrt{neg})$

For 6)

For $f(x) = \dfrac{\sqrt{3-x}}{x^2-4x+3}$

For 6 a) as $x \to 3^-$

the type is "$\frac{0}{0}$" and the function is defined for $x<3$ and $x \neq 1$ we can simplify

$$f(x) = \frac{\sqrt{3-x}}{x^2-4x+3} = \frac{\sqrt{3-x}}{(x-3)(x-1)} = \frac{\sqrt{3-x}}{-(3-x)(x-1)} = \frac{\sqrt{3-x}}{-\sqrt{3-x}\sqrt{3-x}(x-1)} = -\frac{1}{\sqrt{3-x}\,(x-1)}$$

Now the type is "$\frac{-1}{0^+\cdot(2)}$" = "$\frac{1}{0^{--}}$" and the limit is $-\infty$.

For 6b) as $x \to 1^+$

We can use the simplification from above or at least identify the type "$\frac{\sqrt{2}}{0}$" so we have to investigate the sign of the bottom

Since $x>1$ and $f(x) = \dfrac{\sqrt{3-x}}{(x-3)(x-1)}$ we can see that the type is :"$\frac{\sqrt{2}}{(-2)\cdot 0^+}$" = "$\frac{1}{0^-}$" and the limit is $-\infty$.

For 6c) as $x \to +\infty$.

the limit DNE (does not exists) since the function is not defined for big positive x .

For 7)

For $g(x) = \sqrt{3+x}$ and $f(x) = \sqrt{x-5}$

first the domains of the given functions $D_g = [-3, +\infty)$ since $3+x \geq 0$;

$D_f = [5, +\infty)$ since $x-5 \geq 0$.

$f \circ g(x) = f(g(x)) = \sqrt{(..)-5} = \sqrt{\sqrt{3+x}-5}$

we must start in D_g i.e. $x \in [-3, +\infty)$ and we have to guarantee that

$\sqrt{3+x}-5 \geq 0$,so $\sqrt{3+x} \geq 5$

both sides are positive so we can square $(3+x) \geq 25$, and $x \geq 22$,together

$D_{f\circ g} = [22, +\infty)$

$g \circ g(x) = \sqrt{3+(..)} = \sqrt{3+\sqrt{3+x}}$

we must start in $\mathrm{D}_g = [-3, +\infty)$ and quarantee that $3+\sqrt{3+x} \geq 0$

but it is always true for any $x \in [-3, +\infty)$ so $D_{g\circ g} = [-3, +\infty)$.

For 8a) as $x \to 0$

the type is "$\frac{0}{0}$" and if x is a small # ,neg. or pos,$x-3$ is close to -3 so negative and $|x-3| = -(x-3) = 3-x$ $\quad$ $x+3$ is close to 3 so $x+3$ is positive and $|x+3| = x+3$ therefore

$$f(x) = \frac{|x-3|-|x+3|}{x} = \frac{3-x-(x+3)}{x} = \frac{-2x}{x} = -2 \text{ for } x \neq 0$$

so the limit is -2.

ALSO $f(x) =$

$$\frac{|x-3|-|x+3|}{x}\cdot\frac{|x-3|+|x+3|}{|x-3|+|x+3|} = \frac{|x-3|^2-|x+3|^2}{x\cdot(|x-3|+|x+3|)} = \frac{(x-3)^2-(x+3)^2}{x\cdot(|x-3|+|x+3|)} =$$

$$= \frac{x^2-6x+3^2-(x^2+6x+3^2)}{x\cdot(|x-3|+|x+3|)} = \frac{-12x}{x\cdot(|x-3|+|x+3|)} = \frac{-12}{(|x-3|+|x+3|)} \text{ for any } x \neq 0$$

Now the limit is $\quad L = \frac{-12}{3+3} = -2.$

For8b) as $x \to -\infty$

For x big negative number $x-3$ is negative and $|x-3| = -(x-3) = 3-x$,also $x+3$ is negative and

$|x+3| = -(x+3) = -3-x$,

$f(x) = \dfrac{|x-3|-|x+3|}{x} = \dfrac{3-x+x+3}{x} = \dfrac{6}{x}$ so the type of the limit is "$\frac{1}{\infty}$" and $L = 0$.

ALSO for both b) and c)

using the siplification from above $f(x) = \dfrac{-12}{(|x-3|+|x+3|)}$ so the type is "$\frac{-12}{\infty}$" and the limit is 0.

For 8 c) as $x \to +\infty$.

For x big positive both expressions $x-3$ and $x+3$ are positive so we can ignore absolute values and

$f(x) = \dfrac{x-3-(x+3)}{x} = \frac{-6}{x}$ and the type of the limit is "$\frac{-6}{\infty}$" and the limit is 0.

For 9)

For $g(x) = \sqrt{3-x}$ and $f(x) = \dfrac{6}{3x-1}$

first the domains of the given functions $D_f = \left\{x \neq \frac{1}{3}\right\}$ since $3x-1 \neq 0$;

$D_g = (-\infty, 3]$ since $3-x \geq 0$.

$f \circ f(x) = f(f(x)) = \dfrac{6}{3(...)-1} = \dfrac{6}{3 \cdot \frac{6}{3x-1} - 1} = \dfrac{6}{\frac{18-(3x-1)}{3x-1}} = \dfrac{6(3x-1)}{19-3x}$

we must start in D$_f$ i.e. $x \neq \frac{1}{3}$ and we have to guarantee that $19-3x \neq 0$ so $x \neq \frac{19}{3}$ and $D_{f \circ f} = \left\{x \neq \frac{1}{3} \wedge x \neq \frac{19}{3}\right\}$

$g \circ f(x) = \sqrt{3-(..)} = \sqrt{3 - \frac{6}{3x-1}} = \sqrt{\frac{3(3x-1)-6}{3x-1}} = \sqrt{\frac{9x-9}{3x-1}} = 3\sqrt{\frac{x-1}{3x-1}}$

we must start in D$_f$, $x \neq \frac{1}{3}$ and quarantee that $\dfrac{x-1}{3x-1} \geq 0$,split points are $x = 1, \frac{1}{3}$

testing $-^{pos} - -_{\frac{1}{3}} - -^{neg} - -_{1} - -^{pos}-$

so the domain is $D_{g \circ f} = \left(-\infty, \frac{1}{3}\right) \cup [1, +\infty)$