

The University of Calgary
 Department of Mathematics and Statistics
 MATH 249
 Worksheet #3

1. Using the **definition of derivative** find $f'(-1)$ if $f(x) = \frac{4x}{3-x}$.
2. Find y' if $y = (\frac{x^6}{2} - 2x)(4 + \frac{1}{\sqrt{2x}})$ for $x > 0$.
3. Find all points on the graph of $y = \frac{1}{2x^3 + x^2 + 1}$ where the tangent is horizontal.
4. Using the **definition of derivative** find $f'(3)$ if $f(x) = \sqrt{\frac{x}{3} + 3}$.
5. Find $f'(-1)$ if $f(x) = (\frac{x^3}{6} + \frac{1}{2x})(6 + 2x^2)^{\frac{1}{3}}$.
6. Find all points on the graph of $y = \frac{2x}{1+3x}$ where the tangent is parallel to the line $y - 2x = 3$.
7. Using the **definition of derivative** find $f'(\frac{1}{2})$ if $f(x) = 2x - \frac{1}{x}$.
8. Find y' if $y = \sqrt{7x + \frac{3}{x^2} + 4\sqrt{x}}$ for $x > 0$.
9. Find an equation of the tangent line to $y = \frac{2x-3}{4-2x^5}$ at $x = -1$.

Solution

For 1)

$$\begin{aligned} f'(-1) &= \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{\frac{4x}{3-x} - \frac{-4}{4}}{x + 1} = \lim_{x \rightarrow -1} \frac{\frac{4x}{3-x} + 1}{x + 1} = \\ &= \lim_{x \rightarrow -1} \frac{\frac{4x+3-x}{3-x}}{x + 1} = \lim_{x \rightarrow -1} \frac{\frac{3(x+1)}{3-x}}{\frac{x+1}{1}} = \lim_{x \rightarrow -1} \frac{3(x+1)}{(3-x)(x+1)} = \lim_{x \rightarrow -1} \frac{3}{3-x} = \frac{3}{4} \end{aligned}$$

ALSO

$$\begin{aligned} f'(-1) &= \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4(-1+h)}{3-(-1+h)} - \frac{4(-1)}{3+1}}{h} = \\ &= \lim_{h \rightarrow 0} \frac{\frac{4h-4}{4-h} + 1}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{4h-4+4-h}{4-h} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{3h}{4-h} = \lim_{h \rightarrow 0} \frac{3}{4-h} = \frac{3}{4} \end{aligned}$$

(Check by rules $f'(x) = \frac{4}{3-x} + \frac{4x}{(3-x)^2}$ at $x = -1$ $f'(-1) = 1 - \frac{4}{16} = \frac{3}{4}$)

For 2)

use Product Rule $y' = \left(\frac{x^6}{2} - 2x\right)' \left(4 + \frac{1}{\sqrt{2x}}\right) + \left(\frac{x^6}{2} - 2x\right) \left(4 + \frac{x^{-\frac{1}{2}}}{\sqrt{2}}\right)' =$
 now Power Rule $= \left(\frac{1}{2} \cdot 6x^5 - 2\right) \left(4 + \frac{1}{\sqrt{2x}}\right) + \left(\frac{x^6}{2} - 2x\right) \left(0 + \frac{1}{\sqrt{2}} \left(\frac{-1}{2}\right) x^{-\frac{3}{2}}\right)$

so $y' = (3x^5 - 2) \left(4 + \frac{1}{\sqrt{2x}}\right) + \left(\frac{x^6}{2} - 2x\right) \left(\frac{-1}{2\sqrt{2}} x^{-\frac{3}{2}}\right)$ for $x > 0$

we can also multiply out and then differentiate each term using Power Rule
For 3)

slope of a tangent is given by $y' = \left(\frac{1}{2x^3 + x^2 + 1}\right)'$

we can use reciprocal (quotient) or Chain Rule

$$y' = \left([2x^3 + x^2 + 1]^{-1}\right)' = (-1) [2x^3 + x^2 + 1]^{-2} \cdot [2x^3 + x^2 + 1]' =$$

$$= \frac{(-1) [6x^2 + 2x + 0]}{[2x^3 + x^2 + 1]^2}$$

horizontal means slope $m = 0$ $y' = 0$ solve for x

a fraction is 0 only if top is 0 $6x^2 + 2x = 2x(3x + 1) = 0$

$x = 0$ or $x = -\frac{1}{3}$

at $x = 0, y = 1$ and at $x = -\frac{1}{3}$

$$y = \frac{1}{2x^3 + x^2 + 1} \Big|_{x=-\frac{1}{3}} = \left(\frac{-2}{27} + \frac{1}{9} + 1\right)^{-1} = \left(\frac{28}{27}\right)^{-1} = \frac{27}{28}$$

at $(0, 1)$ and at $\left(-\frac{1}{3}, \frac{27}{28}\right)$ the tangent line is horizontal.

For 4)

$$f'(3) = \lim_{x \rightarrow 3} \frac{\sqrt{\frac{x}{3} + 3} - \sqrt{4}}{x - 3} \cdot \frac{\sqrt{\frac{x}{3} + 3} + 2}{\sqrt{\frac{x}{3} + 3} + 2} = \lim_{x \rightarrow 3} \frac{\left(\frac{x}{3} + 3\right) - 4}{x - 3} \cdot \frac{1}{\sqrt{\frac{x}{3} + 3} + 2} =$$

$$= \lim_{x \rightarrow 3} \frac{\frac{x}{3} - 1}{x - 3} \cdot \frac{1}{\sqrt{\frac{x}{3} + 3} + 2} = \lim_{x \rightarrow 3} \frac{\frac{1}{3}(x - 3)}{(x - 3) \left(\sqrt{\frac{x}{3} + 3} + 2\right)} = \frac{1}{3} \cdot \frac{1}{\sqrt{4} + 2} = \frac{1}{12}$$

check by Chain Rule $f'(x) = \left[\left(\frac{1}{3}x + 3\right)^{\frac{1}{2}}\right]' = \frac{1}{2} \left(\frac{1}{3}x + 3\right)^{-\frac{1}{2}} \cdot \frac{1}{3}$

and $f'(3) = \frac{1}{6 \cdot \sqrt{4}} = \frac{1}{12}$

OR

$$f'(3) = \lim_{h \rightarrow 0} \frac{\sqrt{\frac{(3+h)}{3} + 3} - \sqrt{4}}{h} \cdot \frac{\sqrt{\frac{(3+h)}{3} + 3} + 2}{\sqrt{\frac{(3+h)}{3} + 3} + 2} = \lim_{h \rightarrow 0} \frac{\frac{(3+h)}{3} + 3 - 4}{h} \cdot \frac{1}{\sqrt{\frac{(3+h)}{3} + 3} + 2} =$$

$$= \lim_{h \rightarrow 0} \frac{1 + \frac{h}{3} - 1}{h \left(\sqrt{\frac{(3+h)}{3} + 3} + 2\right)} = \frac{\frac{1}{3}}{\sqrt{4} + 2} = \frac{1}{12}$$

For 5)

use Product and then Chain Rules for any $x \neq 0$

$$f'(x) = \left(\frac{x^3}{6} + \frac{1}{2x}\right)' \cdot (6 + 2x^2)^{\frac{1}{3}} + \left(\frac{x^3}{6} + \frac{1}{2x}\right) \cdot \left[(6 + 2x^2)^{\frac{1}{3}}\right]' =$$

$$= \left[\frac{1}{6} \cdot 3x^2 + \frac{1}{2} (-x^{-2})\right] \cdot (6 + 2x^2)^{\frac{1}{3}} + \left(\frac{x^3}{6} + \frac{1}{2x}\right) \cdot \frac{1}{3} (6 + 2x^2)^{-\frac{2}{3}} \cdot (6 + 2x^2)' =$$

$$= \left(\frac{x^2}{2} - \frac{1}{2x^2}\right) \cdot (6 + 2x^2)^{\frac{1}{3}} + \left(\frac{x^3}{6} + \frac{1}{2x}\right) \cdot (6 + 2x^2)^{-\frac{2}{3}} \cdot \frac{4x}{3}$$

then

$$f'(-1) = 0 + \left(\frac{-1}{6} - \frac{1}{2}\right) \left(\frac{-4}{3}\right) 8^{-\frac{2}{3}} = \frac{8}{9} \left(\sqrt[3]{8}\right)^{-2} = \frac{2}{9}.$$

For 6)

by Quotient Rule

$$y' = \frac{(2x)'(1+3x) - 2x(1+3x)'}{(1+3x)^2} = \frac{2(1+3x) - 2x \cdot 3}{(1+3x)^2} = \frac{2}{(1+3x)^2}$$

the slope of a parallel tangent $y' = 2$ (slope of $y = 2x + 3$)

$$\text{solve for } x \quad \frac{2}{(1+3x)^2} = 2 \quad (1+3x)^2 = 1 \quad 1+3x = \pm 1$$

$$\text{OR} \quad 1+6x+9x^2 = 1 \quad 3x(2+3x) = 0$$

$$\text{points are } x = 0, y = 0 \text{ and } x = -\frac{2}{3}, y = \frac{2x}{1+3x} \Big|_{x=-\frac{2}{3}} = \frac{\frac{-4}{3}}{1-2} = \frac{4}{3}$$

at the points $(0, 0)$ and $\left(-\frac{2}{3}, \frac{4}{3}\right)$.

For 7)

$$f\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{2} - \frac{1}{\frac{1}{2}} = 1 - 2 = -1$$

$$f'\left(\frac{1}{2}\right) = \lim_{x \rightarrow \frac{1}{2}} \frac{f(x) - f\left(\frac{1}{2}\right)}{x - \frac{1}{2}} = \lim_{x \rightarrow \frac{1}{2}} \frac{2x - \frac{1}{x} + 1}{x - \frac{1}{2}} = \lim_{x \rightarrow \frac{1}{2}} \frac{\frac{2x^2 - 1 + x}{x}}{\frac{2x - 1}{x}} =$$

$$= \lim_{x \rightarrow \frac{1}{2}} \frac{2(2x^2 + x - 1)}{x(2x - 1)} = \lim_{x \rightarrow \frac{1}{2}} \frac{2(2x - 1)(x + 1)}{x(2x - 1)} = \lim_{x \rightarrow \frac{1}{2}} \frac{2(x + 1)}{x} = 4 \cdot \frac{3}{2} = 6$$

OR

$$f'\left(\frac{1}{2}\right) = \lim_{h \rightarrow 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h} = \lim_{h \rightarrow 0} \frac{2\left(\frac{1}{2} + h\right) - \frac{1}{\frac{1}{2} + h} + 1}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{1 + 2h - \frac{2}{1+2h} + 1}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2(1+h)(1+2h) - 2}{1+2h} \right] =$$

$$= \lim_{h \rightarrow 0} \frac{2}{h} \left[\frac{1 + 3h + 2h^2 - 1}{1+2h} \right] = \lim_{h \rightarrow 0} 2 \left[\frac{3 + 2h}{1+2h} \right] = 6$$

check by Rules $f'(x) = 2 + \frac{1}{x^2}$ so at $x = \frac{1}{2}$ we get 6.

For 8)

$$y = \sqrt{7x + \frac{3}{x^2}} + 4\sqrt{x} = \left(7x + \frac{3}{x^2} + 4\sqrt{x}\right)^{\frac{1}{2}} \text{ by Chain Rule}$$

$$y' = \frac{1}{2} \left(7x + \frac{3}{x^2} + 4\sqrt{x}\right)^{-\frac{1}{2}} \left(7x + 3x^{-2} + 4x^{\frac{1}{2}}\right)' = \frac{7 - 6x^{-3} + 4 \cdot \frac{1}{2}x^{-\frac{1}{2}}}{2\sqrt{7x + \frac{3}{x^2}} + 4\sqrt{x}}$$

$$y' = \frac{7 - \frac{6}{x^3} + \frac{2}{\sqrt{x}}}{2\sqrt{7x + \frac{3}{x^2}} + 4\sqrt{x}} \text{ for } x > 0.$$

For 9)

$$\text{by Quotient Rule} \quad y' = \frac{(2x-3)'(4-2x^5) - (2x-3)(4-2x^5)'}{(4-2x^5)^2}$$

$$\text{so} \quad y' = \frac{(2)(4-2x^5) - (2x-3)(-10x^4)}{(4-2x^5)^2} \text{ at } x = -1$$

slope $m = f'(-1) = \frac{2 \cdot 6 - (-5)(-10)}{36} = \frac{-38}{36} = \frac{-19}{18}$

and $f(-1) = \frac{-5}{6}$ $P\left(-1, -\frac{5}{6}\right)$

so tangent $y = \frac{-19}{18}(x + 1) - \frac{5}{6}$ or $y = \frac{-19}{18}x - \frac{34}{18}$ or $18y + 19x = -34$.