

The University of Calgary
 Department of Mathematics and Statistics
 MATH 249
 Worksheet #4

1. Find an equation of the tangent line to

$$\sqrt{x^2 - y} = \frac{9x}{y} - 1$$

at the point P (5, 9).

2. Find a general antiderivative of $f(x) = \frac{5\sqrt{x} - 6x^3 - 8x^2 + 3}{x^2}$ for $x > 0$.

3. Solve $y'' = 2 \sin(\pi - 2x)$ with $y'(\pi) = 0$ and $y(\pi) = 3$.

4. Find the second derivative of $f(x) = x \cos(x^2)$. Simplify.

5. Find y' in terms of x and y if $2x + 3y = \frac{y^2}{x + 1}$.

6. Find a general antiderivative of $f(x) = \frac{1}{\cos^2(3x - 1)}$ in the domain (find the domain).

7. Solve $y'' = \frac{3}{\sqrt{x}} - 6x$, $y'(4) = 2$, $y(4) = 0$.

8. Find the second derivative of $y = \frac{1}{1 + x^2}$. Simplify.

9. Find an equation of the tangent line at the point $(6, \pi)$ to

$$2 \cos \frac{y}{x} + \frac{xy}{6} = \sqrt{3} + \pi.$$

10. Solve (i.e. find y including an interval)

$$y' = \frac{1}{(5 - x)^3}$$

with $y(4) = 1$

11. Find $\int \left(3\sqrt{x} - \frac{1}{3x} \right)^2 dx$ for $x > 0$.

SOLUTIONS

For 1)

Use implicit differentiation and Chain Rule on the left ,Quotient Rule on right:

$$\frac{1}{2} (x^2 - y)^{-\frac{1}{2}} \cdot (x^2 - y)' = 9 \cdot \left(\frac{x}{y} \right)'$$

$$\frac{1}{2} (x^2 - y)^{-\frac{1}{2}} \cdot (2x - y') = 9 \cdot \frac{1 \cdot y - x \cdot y'}{y^2}$$

now, $x = 5, y = 9, y' = m$

$$\frac{1}{2} (25 - 9)^{-\frac{1}{2}} (10 - m) = 9 \cdot \frac{9 - 5m}{9^2} \text{ so } \frac{1}{8} (10 - m) = \frac{1}{9} (9 - 5m)$$

multiply by $9 \cdot 8$

$90 - 9m = 72 - 40m$ thus $31m = -18$ and
 $m = -\frac{18}{31}$ and an equation is $y = -\frac{18}{31}(x - 5) + 9$.

For 2)

get rid of the quotient

$$\int f(x)dx = 5 \int \frac{\sqrt{x}}{x^2} dx - 6 \int \frac{x^3}{x^2} dx - 8 \int \frac{x^2}{x^2} dx + 3 \int x^{-2} dx =$$

$$5 \int x^{-\frac{3}{2}} dx - 6 \int x dx - 8 \int dx + 3 \int x^{-1} dx + c = 5 \cdot (-2) x^{-\frac{1}{2}} - 6 \cdot \frac{x^2}{2} - 8x + 3 \cdot \frac{x^{-1}}{-1} + c$$

$$= -\frac{10}{\sqrt{x}} - 3x^2 - 8x - \frac{3}{x} + c \text{ for } x > 0.$$

For 3)

$$y' = \int y'' dx = 2 \int \sin(\pi - 2x) dx = 2 \cdot \frac{-\cos(\pi - 2x)}{-2} + c_1 = \cos(\pi - 2x) + c_1$$

$$\text{using } \int \sin(ax + b) dx = \frac{-\cos(ax + b)}{a} + c, a = -2, b = \pi$$

now use the condition $y' = 0$ for $x = \pi$

$$0 = \cos(-\pi) + c_1 = -1 + c_1 \quad \text{so } c_1 = 1 \text{ and } y' = \cos(\pi - 2x) + 1$$

$$\text{using } \int \cos(ax + b) dx = \frac{\sin(ax + b)}{a} + c$$

$$y = \int y' dx = \int \cos(\pi - 2x) dx + \int 1 dx + c_2 = \frac{\sin(\pi - 2x)}{-2} + x + c_2 = -\frac{1}{2} \sin(\pi - 2x) + x + c_2$$

use the second condition $y = 3$ for $x = \pi$

$$3 = -\frac{1}{2} \sin(-\pi) + \pi + c_2 = 0 + \pi + c_2 \text{ so } c_2 = 3 - \pi$$

and the solution is

$$y = -\frac{1}{2} \sin(\pi - 2x) + x + 3 - \pi$$

For 4)

by Product and Chain Rules

$$f'(x) = [x \cos(x^2)]' = (x)' \cdot \cos(x^2) + x (-\sin(x^2)) (x^2)' = \cos(x^2) - 2x^2 \sin(x^2)$$

again

$$f''(x) = [\cos(x^2) - 2x^2 \sin(x^2)]' = -2x \sin(x^2) - 4x \sin(x^2) - 2x^2 \cos(x^2) 2x =$$

$$= -6x \sin(x^2) - 4x^3 \cos(x^2)$$

For 5)

use implicit differentiation, Quotient and Chain Rules:

$$(2x + 3y)' = \left(\frac{y^2}{x+1} \right)'$$

$$2 + 3y' = \frac{2yy'(x+1) - y^2}{(x+1)^2} \text{ multiply both side by } (x+1)^2$$

$$2(x+1)^2 + 3y'(x+1)^2 = 2yy'(x+1) - y^2 \text{ all terms with } y'$$

$$y' [3(x+1)^2 - 2y(x+1)] = -y^2 - 2(x+1)^2$$

so

$$y' = \frac{-y^2 - 2(x+1)^2}{3(x+1)^2 - 2y(x+1)} \text{ if the denominator is not 0.}$$

OR

we can simplify first by multiplying by $(x+1)$

$$2x + 3y = \frac{y^2}{x+1} \quad (2x + 3y)(x+1) = y^2$$

then

$2x^2 + 2x + 3xy + 3y = y^2$ then differentiate by Pr. and Chain rules:

$$4x + 2 + 3y + 3xy' + 3y' = 2yy' \quad 4x + 2 + 3y = y'(2y - 3x - 3)$$

then

$$y' = \frac{4x + 2 + 3y}{2y - 3x - 3} \text{ if the denominator is not 0.}$$

Notice that it looks different because we have a relation between x and y .

For 6)

$$\int \frac{1}{\cos^2(3x-1)} dx = \frac{1}{3} \tan(3x-1) + c$$

$$\text{since } (\tan 3x - 1)' = \sec^2(3x - 1) \cdot 3 = \frac{3}{\cos^2(3x - 1)}$$

for $3x - 1 \neq \frac{\pi}{2} + k\pi$ so $x \neq \frac{1}{3} + \frac{\pi}{6} + k\frac{\pi}{3}, k = 0, \pm 1, \pm 2, \pm 3, \dots$

For 7)

$$y'' = \frac{3}{\sqrt{x}} - 6x \quad y'(4) = 2 \quad y(4) = 0.$$

$$y' = \int y'' dx = 3 \int x^{-\frac{1}{2}} dx - 6 \int x dx + c_1 = 6\sqrt{x} - 3x^2 + c_1$$

now $x = 4$ $y' = 2$ solve for c_1 :

$$2 = 6 \cdot 2 - 3 \cdot 16 + c_1 \quad c_1 = 38$$

so

$$y' = 6\sqrt{x} - 3x^2 + 38 \quad \text{for } x > 0$$

again

$$y = \int y' dx = 6 \int x^{\frac{1}{2}} dx - 3 \int x^2 dx + 38 \int dx = 6 \cdot \frac{2}{3} x^{\frac{3}{2}} - 3 \cdot \frac{x^3}{3} + 38x + c_2$$

$$y = 4x^{\frac{3}{2}} - x^3 + 38x + c_2$$

now $x = 4$ $y = 0$ solve for c_2 :

$$0 = 4 \cdot 2^3 - 4^3 + 38 \cdot 4 + c_2 \quad c_2 = -4(8 - 16 + 38) = -120$$

thus the solution of the given problem is

$$y = 4x^{\frac{3}{2}} - x^3 + 38x - 120 \quad \text{for any } x > 0.$$

For 8)

by Chain Rule

$$y' = \left(\frac{1}{1+x^2} \right)' = \left[(1+x^2)^{-1} \right]' = (-1)(1+x^2)^{-2} 2x = -2x(1+x^2)^{-2}$$

by product and chain rules

$$y'' = (-2x)'(1+x^2)^{-2} - 2x[(1+x^2)^{-2}]' = -2(1+x^2)^{-2} - 2x(-2)(1+x^2)^{-3} 2x = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$$

Or by Quotient Rule

$$y' = \left(\frac{1}{1+x^2} \right)' = \frac{0-2x}{(1+x^2)^2} \quad y'' = \left(\frac{-2x}{(1+x^2)^2} \right)' = \frac{-2(1+x^2)^2 + 2x2(1+x^2)2x}{(1+x^2)^4} = \frac{(1+x^2)[-2(1+x^2) + 8x^2]}{(1+x^2)^4} = \frac{-2+6x^2}{(1+x^2)^3}$$

For 9)

Use implicit differentiation: $2 \left[\cos \frac{y}{x} \right]' + \frac{1}{6} (xy)' = (\sqrt{3} + \pi)'$

$$2 \left(-\sin \frac{y}{x} \right) \left(\frac{y}{x} \right)' + \frac{1}{6} (1 \cdot y + x \cdot y') = 0$$

$$-2 \sin \frac{y}{x} \cdot \frac{y' \cdot x - y \cdot 1}{x^2} + \frac{1}{6} (y + xy') = 0$$

Now $x = 6, y = \pi,$ and $y' = m$:

$$-2 \sin \frac{\pi}{6} \cdot \frac{6m - \pi}{36} + \frac{1}{6} (\pi + 6m) = 0, \text{ multiply both sides by } 36 \text{ and use } \sin \frac{\pi}{6} = \frac{1}{2}$$

thus

$$-(6m - \pi) + 6(\pi + 6m) = 0 \text{ and the equation is now: } -6m + \pi + 6\pi + 36m = 0,$$

thus

$$30m = -7\pi \text{ and } m = -\frac{7\pi}{30}. \text{ The equation of the tangent line is :}$$

$$y = -\frac{7\pi}{30}(x - 6) + \pi$$

For 10)

For $x \neq 5$ $y = \int y' dx = \int (5-x)^{-3} dx = \frac{(5-x)^{-2}}{-2(-1)} + c = \frac{1}{2(5-x)^2} + c$

using $\int (ax+b)^r dx = \frac{(ax+b)^{r+1}}{a(r+1)} + c$, where $a = -1, b = 5, r = -3$

now if $x = 4, y = 1$ solve for c : $1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$.

Together the solution is $y = \frac{1}{2}(5-x)^{-2} + \frac{1}{2}$ for $x \in (-\infty, 5)$
since the condition is at $x = 4 < 5$.

For 11)

$$\int \left(3\sqrt{x} - \frac{1}{3x} \right)^2 dx$$

(get rid of the power using $(A - B)^2 = A^2 - 2AB + B^2$)

$$= \int \left[(3\sqrt{x})^2 - 2 \cdot 3\sqrt{x} \cdot \frac{1}{3x} + \left(\frac{1}{3x} \right)^2 \right] dx =$$

$$= 9 \int x dx - 2 \int x^{-\frac{1}{2}} dx + \frac{1}{9} \int x^{-2} dx = 9 \cdot \frac{1}{2} x^2 - 2 \cdot 2x^{\frac{1}{2}} + \frac{1}{9} \cdot \frac{x^{-1}}{-1} + c = \frac{9}{2} x^2 - 4\sqrt{x} - \frac{1}{9x} + c$$

for $x > 0$.