THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL Handout
MATH 249-01

1. Evaluate the limits:

for a)

Since $\cos\left(\frac{\pi}{2}\right) = 0$ the type is "$\frac{0}{0}$" so we can use L'Hopital Rule

$$\lim_{x\to\pi} \frac{\cos\left(\frac{x}{2}\right)}{\pi - x} = \lim_{x\to\pi} \frac{-\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}}{-1} = \frac{1}{2} \qquad (\sin\tfrac{\pi}{2} = 1).$$

for b)

The type is "$\frac{\text{DNE}}{\infty}$" but $-1 \le \cos\frac{x}{2} \le 1$ and $\pi - x > 0$

so $$\frac{-1}{\pi - x} \le \frac{\cos\frac{x}{2}}{\pi - x} \le \frac{1}{\pi - x}$$

Since both $\lim_{x\to-\infty} \frac{\pm 1}{\pi - x} = 0$ by Squeeze Theorem $\lim_{x\to-\infty} \frac{\cos\left(\frac{x}{2}\right)}{\pi - x} = 0.$

for c)

$$\lim_{x\to 0^+} x\ln x = \text{"}0^+(-\infty)\text{"} = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \text{"}\frac{-\infty}{+\infty}\text{"}(\text{L'H.}) = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$$

2. Find the domain and the derivative of

(a) $f(x) = \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)}$

domain is $D = \{x \ne 0\} = (-\infty, 0) \cup (0, +\infty)$

by Product and Chain Rules $f'(x) = \left(\frac{x}{3}\right)' e^{-\sin\left(\frac{3}{x}\right)} + \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \left(-\sin\frac{3}{x}\right)' =$

$$= \frac{1}{3} e^{-\sin\left(\frac{3}{x}\right)} + \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \left(-\cos\tfrac{3}{x}\right)\left(-\frac{3}{x^2}\right) = e^{-\sin\left(\frac{3}{x}\right)} \left(\frac{1}{3} + \frac{1}{x}\cos\tfrac{3}{x}\right)$$

OR use log.diff $\ln f = \ln\frac{x}{3} + \ln e^{-\sin\frac{3}{x}} = \ln x - \ln 3 - \sin\frac{3}{x}$

then $$\frac{f'}{f} = \frac{1}{x} - 0 - \cos\tfrac{3}{x}\cdot(3x^{-1})' = \frac{1}{x} - \cos\tfrac{3}{x}\cdot(-3)\,x^{-2} = \frac{1}{x} + 3x^{-2}\cos\tfrac{3}{x}$$

so $f'(x) = \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \left[\frac{1}{x} + \frac{3}{x^2}\cos\frac{3}{x}\right] =$ ..... as above.

(b) $f(x) = \dfrac{\ln(2x-3)}{e^{-x^2}}$

for the domain $2x - 3 > 0$ so $x > \frac{3}{2}$ and $D = \left(\frac{3}{2}, +\infty\right)$

we can change the function to a product

$f(x) = e^{x^2}\cdot\ln(2x-3)$ then by Product and Chain Rules

$$f'(x) = e^{x^2}(x^2)'\ln(2x-3) + e^{x^2}\cdot\frac{1}{2x-3}(2x-3)'$$

$$= e^{x^2}(2x)\ln(2x-3) + e^{x^2}\cdot \tfrac{1}{2x-3}\cdot 2 = 2e^{x^2}\left(x\ln(2x-3) + \frac{1}{2x-3}\right)$$

3. **A**

Given $\quad y = \dfrac{1}{x^2+x-2}$ if $y' = -\dfrac{2x+1}{(x^2+x-2)^2}$ and $y'' = \dfrac{6x^2+6x+6}{(x^2+x-2)^3}$ i.e.

part a)

domain $D = (-\infty,-2)\cup(-2,1)\cup(1,+\infty)$

since the denominator $x^2+x-2 = (x+2)(x-1)$ cannot be 0

$x=-2$ and $x=1$ are V.A.(vertical asymptotes)

since $\displaystyle\lim_{x\to -2^+}\frac{1}{(x+2)(x-1)} = \text{"}\tfrac{1}{0^+\cdot(-3)}\text{"} = -\infty$

$\displaystyle\lim_{x\to -2^-}\frac{1}{(x+2)(x-1)} = \text{"}\tfrac{1}{0^-(-3)}\text{"} = +\infty$ and $\displaystyle\lim_{x\to 1^+}\frac{1}{(x+2)(x-1)} = \text{"}\tfrac{1}{3\cdot 0^+}\text{"} = +\infty$

$\displaystyle\lim_{x\to 1^-}\frac{1}{(x+2)(x-1)} = \text{"}\tfrac{1}{0^-}\text{"} = -\infty$

No x-intercepts since never $y=0$, for $x=0 \qquad y=-\frac{1}{2}$

For horizontal asymptotes $\displaystyle\lim_{x\to\pm\infty}\frac{1}{x^2+x-2} = \text{"}\tfrac{1}{\infty}\text{"} = 0$

thus $y=0$ is H.A. at both ends.We will go back to the range later.

part b)

given $y' = -\dfrac{2x+1}{(x^2+x-2)^2}$ so obviously $x=-2,1$ are singular points

and for critical points solve $y'=0$

so $2x+1=0$ thus $x=-\frac{1}{2}$ is a critical point and $f(-\frac{1}{2}) = \left[\frac{1}{4}-\frac{1}{2}-2\right]^{-1} = -\frac{4}{9}$

testing $\quad y' \quad -^{+} -^{+} -^{+} -_{-2} -^{+} -^{+} -_{-\frac{1}{2}} -^{-} -^{-} -_{1} -^{-} -^{-} -^{-} -$

$\qquad y \quad - -^{incr} - -_{-2} -^{incr} - -_{-\frac{1}{2}} - -^{decr} - -_{1} - -^{decr} --$

and the function is **decreasing** on the intervals $\left(-\frac{1}{2},1\right)$ and $(1,+\infty)$,

it is **increasing** on $(-\infty,-2)$ and on $\left(-2,-\frac{1}{2}\right)$

local maximum at $x = -\dfrac{1}{2} \qquad y = \dfrac{-4}{9}$

part c)

given $y'' = \dfrac{6x^2+6x+6}{(x^2+x-2)^3} = \dfrac{6(x^2+x+1)}{(x+2)^3(x-1)^3}$ for $x\neq -2,1$

solve for possible inflection points solve $\qquad y''=0$

never $\quad x^2+x+1=0$ since the discriminant $D=1-4=-3$ is negative

always $x^2+x+1>0$,only singular points testing $\quad y'' \quad -^{+} -^{+} -_{-2} -^{-} -^{-} -^{-} -^{-} -_{1} -^{+} -^{+} -$

$y \quad -^{conc.up} - -_{-2} - -^{conc.down} - - -_{1} -^{conc.up} -^{+} -$

therefore the function is **concave up** on $(-\infty,-2)$ and on $(1,+\infty)$

and it is **concave down** on $(-2,1)$. TOGETHER

from the graph we can read the range $R=\left(-\infty,-\frac{4}{9}\right] \cup (0,+\infty)$

THERE IS A GAP IN THE RANGE !

3B.

1.step function is a polynomial so $D=(-\infty,+\infty)$

"ends": $\lim\limits_{x\to\infty} x(4-x)^3 = +\infty\cdot(-\infty) = -\infty$, $\lim\limits_{x\to-\infty} x(4-x)^3 = -\infty\cdot(+\infty) = -\infty$

No V. or H. asymptotes;when $x=0,4$ $\quad y=0$;

2.step

by Product Rule $y' = 1\cdot(4-x)^3 + x\cdot 3(4-x)^2\cdot(-1) = (4-x)^2(4-x-3x) = 4(4-x)^2(1-x)$

solve $y'=0$ for critical points :$x=1,4$

testing $\quad y' -^{+} -^{+} -^{+} -_{1} -^{-} -^{-} -^{-} -_{4} -^{-} -^{-} -^{-} -$

$y \quad - -^{incr.} - -_{1} - -^{decr} - -_{4} - -^{decr} - -$

the function is **incr**.on$(-\infty,1)$ and **decr** on $(1,+\infty)$ but has a horizontal tangents at $x=1$ and $x=4$.

also $f(1)=27$ and $f(4)=0$,and $x=1$ is a local maximum

3.step

$y'' = 4\cdot 2(4-x)(-1)(1-x) + 4(4-x)^2(-1) = -4(4-x)(2-2x+4-x) =$

$= -4(4-x)(6-3x) = 12(x-4)(2-x)$

for possible inflection points solve $y''=0$ $\quad x=4,2$

testing $\quad y'' -^{-} -^{-} -^{-} -_{2} -^{+} -^{+} -_{4} -^{-} -^{-} -^{-} -$

$y \quad - -^{conc.down} - -_{2} -^{conc.up} - -_{4} - -^{conc.down} - -$

therefore the function is **concave up** on $(2,4)$ and concave down on $(-\infty,2)$ and on $(4,+\infty)$

TOGETHER we can see that the **range** is $-(\infty,27]$ and 27 is abs.maxium value at $x=1$.

4. For $f(x) = \dfrac{1}{\sqrt{2x^2+1}}$

find $f(2) = \frac{1}{\sqrt{9}} = \dfrac{1}{3}$ and $f'(x) = \frac{-1}{2}(2x^2+1)^{-\frac{3}{2}}\cdot 4x = -2x\cdot\left(\frac{1}{\sqrt{2x^2+1}}\right)^3$,

$f'(2) = -4\cdot\left(\frac{1}{3}\right)^3 = \dfrac{-4}{27}$ so the linearization is $L(x) = \dfrac{1}{3} - \dfrac{4}{27}(x-2)$ and

$$\frac{1}{\sqrt{2x^2+1}} \mathrel{\mathring{=}} \frac{1}{3} - \frac{4}{27}(x-2) \qquad \text{for } x \text{ close to } 2.$$

To get $\frac{1}{\sqrt{3}}$ we need $2x^2+1=3$ so $x=\pm 1$ but $x=1$ is closer to 2

so substitute $x=1$: $\quad \frac{1}{\sqrt{3}} \doteq \frac{1}{3} + \frac{4}{27} = \frac{13}{27} = 0.481.$

5. see graphs

6. $\mathbb{A}$

Dimensions: $\quad x \times x \times y$

so the volume $V = x^2 y = 18$ .....given

looking for min of cost $C = 2\cdot$area of the base and lid $+3\cdot$ area of sides$= 2\cdot 2x^2 + 3\cdot 4xy$

so $\qquad C = 4x^2 + 12xy$

reduce to one variable: $y = \dfrac{18}{x^2}$ back to the cost $C(x) = 4x^2 + 12x\cdot\dfrac{18}{x^2} = 4\left(x^2 + 3\cdot 18x^{-1}\right)$

for critical points $C'(x) = 4\left(2x - 3\cdot 2\cdot 9x^{-2}\right) = 8\cdot\dfrac{x^3-27}{x^2} = 0$ means $x^3 = 27, x = 3$ C.P.

to justify that we have found **minimum**

$C''(x) = 4\left(2 + 6\cdot 18x^{-3}\right) > 0$ for $x > 0$,so the function is conc.up

the solution is $x = 3m, y = \frac{18}{3} = 6m$

B

Draw some rectangles and name the dimensions $x \times y$

given $A = xy = 280$ $\qquad$ and we are looking for minimum of the cost

$C = 25(x+2y) + 10x = 35x + 50y$

reduce to one variable : $\quad y = \dfrac{280}{x} \quad$ thus $\quad f(x) = 35x + \frac{7\cdot 40\cdot 50}{x} = 35\left(x + \dfrac{400}{x}\right)$

and $\quad f'(x) = 35(1 - \dfrac{400}{x^2}) \qquad$ Solve $f' = 0 \qquad x^2 = 400 \quad$ and $x = 20$m $\quad (x > 0)$

back to $y = \frac{280}{20} = 14$m

To justify that we have found minimum use $f''(x) = \frac{35\cdot 800}{x^3} > 0$ for $x > 0$

so f is concave up and the critical point is a minimum.

(OR $f' > 0$ for $x > 20$ and $f' < 0$ for $x < 20$)

Thus the dimensions are 20m$\times$14m with one of the longer sides to be fenced.

7. Evaluate:

(a) for $x > 0$

$\int \frac{3\sqrt{x}-5}{x\sqrt{x}}dx = \int \frac{3\sqrt{x}}{x\sqrt{x}}dx - \int \frac{5}{x\sqrt{x}}dx = 3\int \frac{1}{x}dx - 5\int x^{-\frac{3}{2}}dx =$

$= 3\ln|x| - 5\cdot(-2)x^{-\frac{1}{2}} + c = 3\ln x + \frac{10}{\sqrt{x}} + c$

(b) $\int 2x^3\sqrt{2x^2+3}dx$

by substitution $u = 2x^2+3 \quad du = 4xdx \quad \frac{1}{2}du = 2xdx$

re-arange the integral $= \int x^2\sqrt{2x^2+3}\cdot 2xdx = \frac{1}{2}\int (?)\sqrt{u}du = .$

from the substitution $\quad \frac{u-3}{2} = x^2$ so

the integral $= \frac{1}{2}\int \frac{u-3}{2}\sqrt{u}du = \frac{1}{4}\int (u-3)\sqrt{u}du = \frac{1}{4}\int u^{\frac{3}{2}}du - \frac{3}{4}\int u^{\frac{1}{2}}du =$

using $\int x^r dx = \frac{x^{r+1}}{r+1} + c$ for $r \neq -1$

$=\frac{1}{4}\cdot\frac{2}{5}u^{\frac{5}{2}} - \frac{3}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} + c =$(back to x)$= \frac{1}{10}(2x^2+3)^{\frac{5}{2}} - \frac{1}{2}(2x^2+3)^{\frac{3}{2}} + c$ for any x.

8. (a) $\int_{\frac{1}{2}}^{1} \frac{3^{\frac{1}{x}}}{x^2}dx$

by substitution $u = \frac{1}{x} \quad du = -\frac{1}{x^2}dx \quad -du = \frac{1}{x^2}dx$ and

x	u
$\frac{1}{2}$	2
1	1

the integral$= -\int_2^1 3^u du = \left[\frac{3^u}{\ln 3}\right]_1^2 = \frac{1}{\ln 3}\left[3^2 - 3\right] = \frac{6}{\ln 3}$

using $\int a^x dx = \frac{1}{\ln a}a^x$ for $a > 0, a \neq 1$

(b) $\int_0^1 \frac{4x+3}{3-2x}dx$

by substitution $u = 3-2x \quad du = -2dx \quad -\frac{1}{2}du = dx$ and

x	u
0	3
1	1

the integral $=-\frac{1}{2}\int_3^1 \frac{?}{u}du = \frac{1}{2}\int_1^3 \frac{?}{u}du =$ from the substitution

$2x = 3-u$ so $\quad 4x = 6-2u$ and $\quad 4x+3 = 9-2u$

therefore the integral $=\frac{1}{2}\int_1^3 \frac{9-2u}{u}du = \frac{9}{2}\int_1^3 \frac{1}{u}du - \int_1^3 du = \frac{9}{2}\left[\ln|u|\right]_1^3 - [3-1] =$

$= \frac{9}{2}\ln 3 - 2$

9. **a)**

simplify first $\quad \ln(x+1)^2 = \ln(4x) \quad$ then apply exp. f.

$(x+1)^2 = 4x \quad x^2 - 2x + 1 = 4x$

$x^2 - 2x + 1 = 0 \quad (x-1)^2 = 0$

$x = 1$ is the solution.

b)

cross multiply first $\quad 4^{3x} = 5 \cdot 2^{2x+1}$, then apply ln to both sides

thus

$3x \ln 4 = \ln 5 + (2x+1) \ln 2 \qquad 3x \ln 4 - 2x \ln 2 = \ln 5 + \ln 2 = \ln(5 \cdot 2)$

$3x \ln 2^2 - 2x \ln 2 = (6x - 2x) \ln 2 = 4x \ln 2 = \ln 10$

or $\quad x(3 \ln 4 - 2 \ln 2) = x \ln \frac{4^3}{2^2} = x \ln 16$

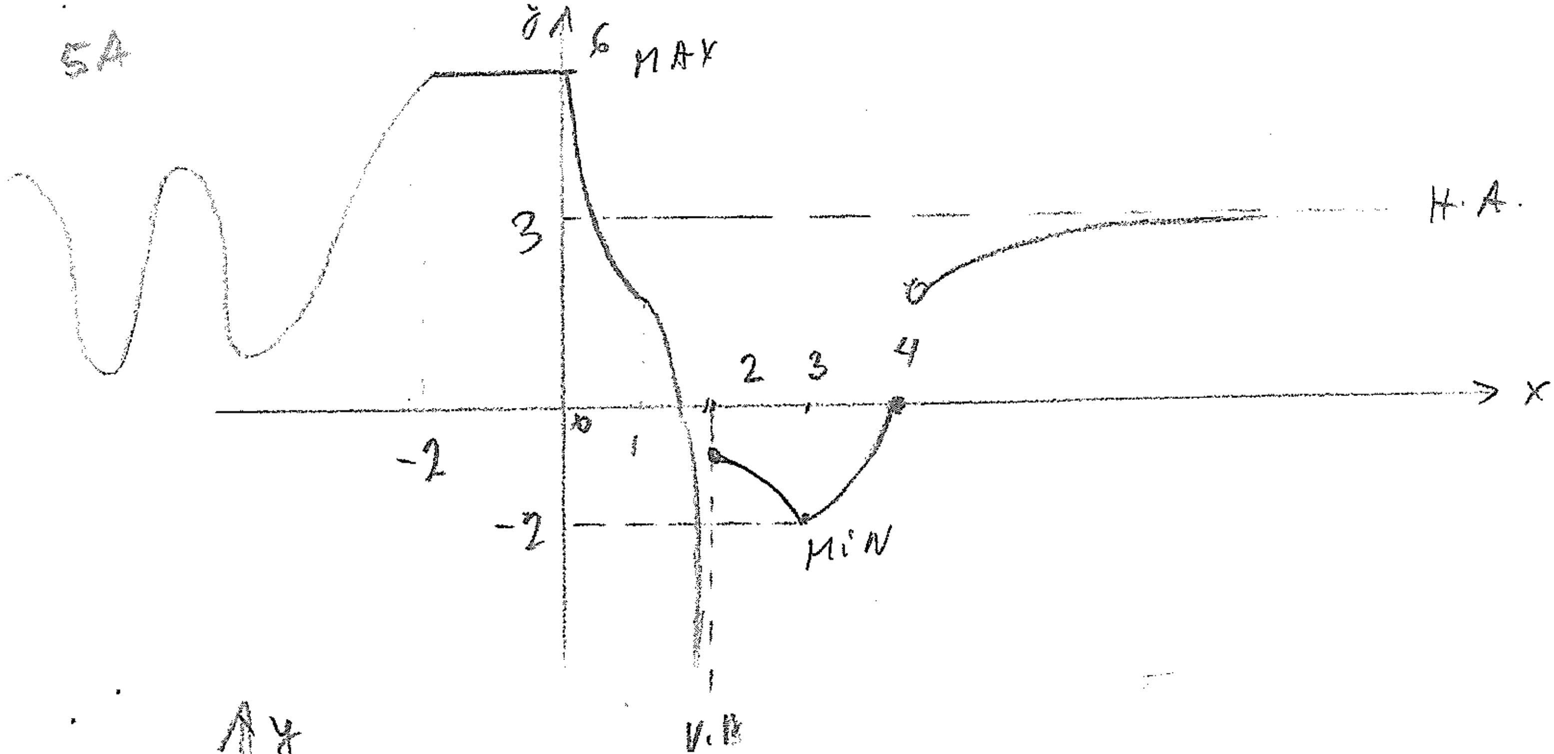
and finally $\quad x \ln 2^4 = \ln 10 \qquad x = \dfrac{\ln 10}{\ln 16} = \dfrac{\ln 10}{4 \ln 2}$

OR

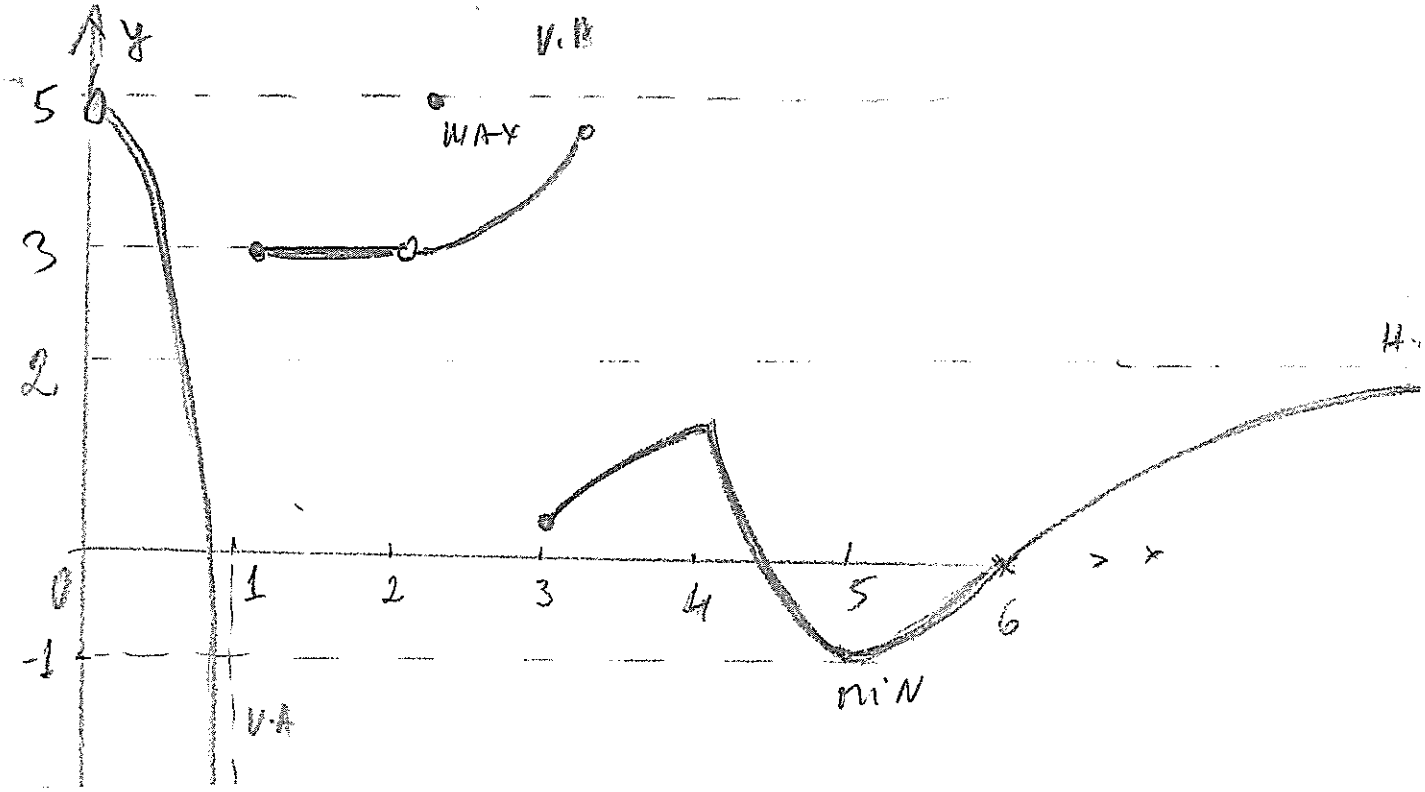
$$\frac{4^{3x}}{2^{2x+1}} = \frac{2^{6x}}{2^{2x+1}} = 2^{4x-1} = 5 \qquad 2^{4x} = 10$$

and then apply $\log_2$ to both sides: $\quad 4x = \log_2 10 \qquad x = \dfrac{1}{4} \log_2 10.$

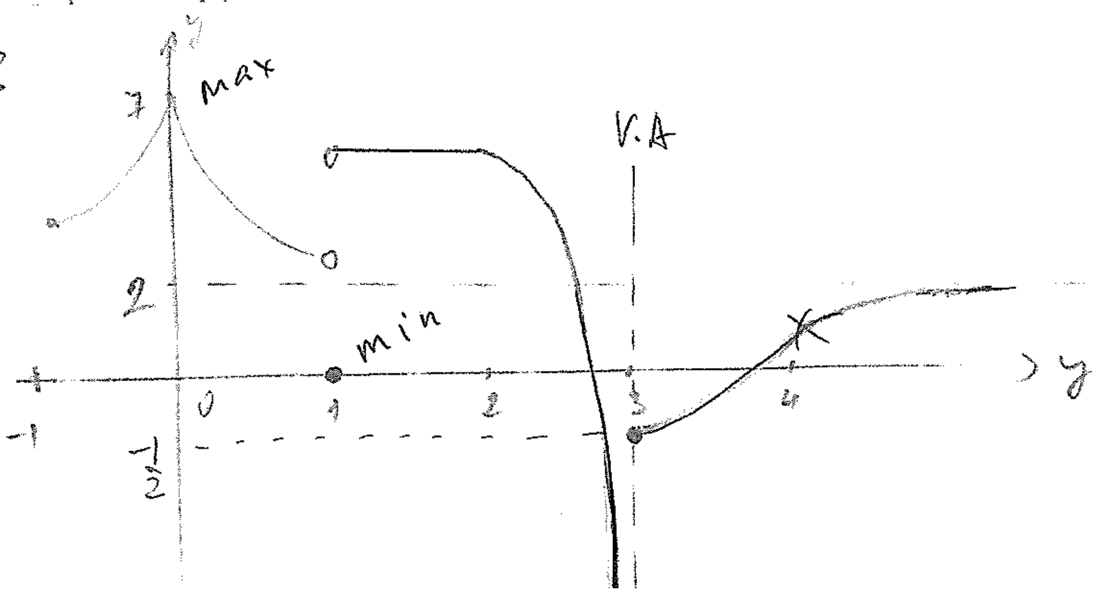
5A



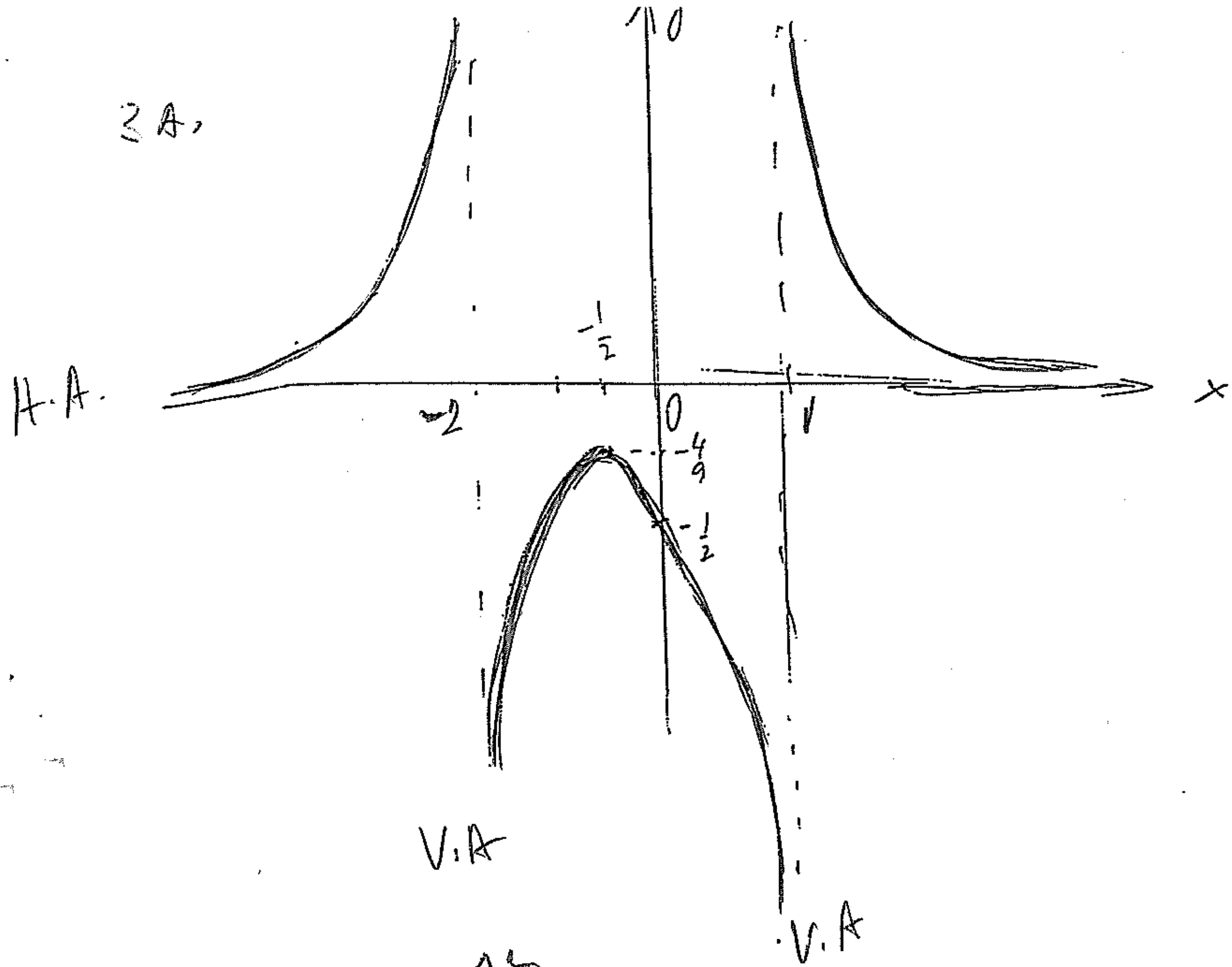
5B



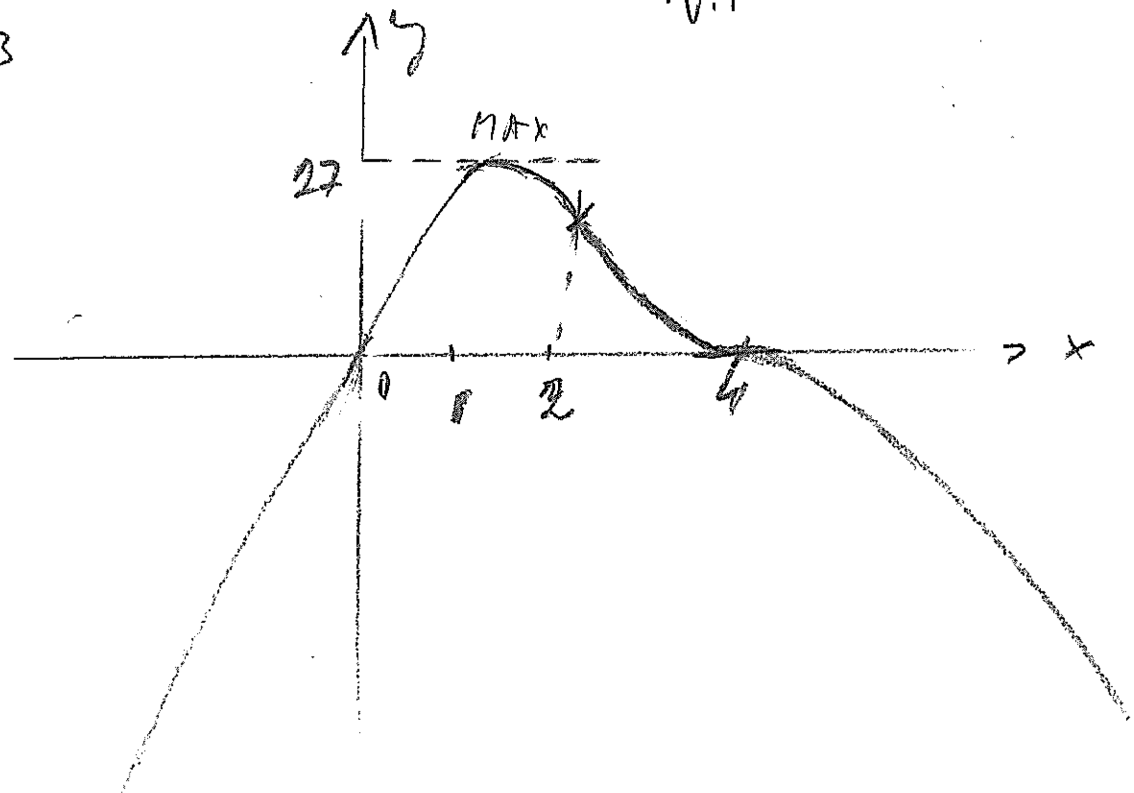
5C



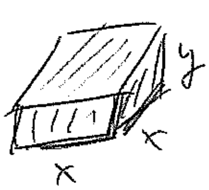
3A.



3B



6.A



6B.