Solutions for Worksheet 5a

1. (a) $f(x) = \sqrt{2-3x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2-3(x+h)} - \sqrt{2-3x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{2-3(x+h)} - \sqrt{2-3x}\right)\left(\sqrt{2-3(x+h)} + \sqrt{2-3x}\right)}{h\left(\sqrt{2-3(x+h)} + \sqrt{2-3x}\right)}$$

$$= \lim_{h\to 0} \frac{2-3(x+h) - (2-3x)}{h\left(\sqrt{2-3(x+h)} + \sqrt{2-3x}\right)}$$

$$= \lim_{h\to 0} \frac{-3h}{h\left(\sqrt{2-3(x+h)} + \sqrt{2-3x}\right)}$$

$$= \lim_{h\to 0} \frac{-3}{\sqrt{2-3(x+h)} + \sqrt{2-3x}} = \frac{-3}{2\sqrt{2-3x}}$$

(b) $f(x) = \sqrt{4x+1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{4(x+h)+1} - \sqrt{4x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{4(x+h)+1} - \sqrt{4x+1}\right)\left(\sqrt{4(x+h)+1} + \sqrt{4x+1}\right)}{h\left(\sqrt{4(x+h)+1} + \sqrt{4x+1}\right)}$$

$$= \lim_{h\to 0} \frac{4(x+h)+1-(4x+1)}{h\left(\sqrt{4(x+h)+1} + \sqrt{4x+1}\right)}$$

$$= \lim_{h\to 0} \frac{4h}{h\left(\sqrt{4(x+h)+1} + \sqrt{4x+1}\right)} =$$

$$= \lim_{h\to 0} \frac{4}{\sqrt{4(x+h)+1} + \sqrt{4x+1}} = \frac{4}{2\sqrt{4x+1}}$$

$$= \frac{2}{\sqrt{4x+1}}$$

(c) $f(x) = x^3 - 2x.$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - 2(x+h) - [x^3 - 2x]}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 2h}{h}$$

$$= \lim_{h\to 0} \left(3x^2 + 3xh + h^2 - 2\right)$$

$$= 3x^2 - 2.$$

(d) $f(x) = 3x^2 - 4x + 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3(x+h)^2 - 4(x+h) + 1 - (3x^2 - 4x + 1)}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2+2hx+h^2) - 4x - 4h + 1 - (3x^2 - 4x + 1)}{h}$$

$$= \lim_{h\to 0} \left(\frac{6hx + 3h^2 - 4h}{h}\right)$$

$$= \lim_{h\to 0} \left(6x + 3h - 4\right) = 6x - 4$$

(e) $$f(x) = \frac{1-x}{2x+1}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1-(x+h)}{2(x+h)+1} - \frac{1-x}{2x+1}}{h}$$

$$= \lim_{h\to 0} \frac{(1-x-h)(2x+1) - (1-x)[2x+2h+1]}{h(2x+1)[2(x+h)+1]}$$

$$= \lim_{h\to 0} \frac{(1-x)(2x+1) - h(2x+1) - [(1-x)(2x+1) + 2h(1-x)]}{h(2x+1)(2x+2h+1)}$$

$$= \lim_{h\to 0} \frac{-h(2x+1) - 2h(1-x)}{h(2x+1)(2x+2h+1)}$$

$$= \lim_{h\to 0} \frac{-3h}{h(2x+1)(2x+2h+1)}$$

$$= \lim_{h\to 0} \frac{-3}{(2x+1)(2x+2h+1)}$$

$$= \frac{-3}{(2x+1)^2}$$

(f) $$f(x) = \frac{3-4x}{5x+4}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{3-4(x+h)}{5(x+h)+4} - \frac{3-4x}{5x+4}}{h}$$

$$= \lim_{h\to 0} \frac{(3-4x-4h)(5x+4) - (3-4x)(5x+5h+4)}{h(5x+5h+4)(5x+4)}$$

$$= \lim_{h\to 0} \frac{(3-4x)(5x+4) - 4h(5x+4) - (3-4x)(5x+4) - 5h(3-4x)}{h(5x+5h-4)(5x+4)}$$

$$= \lim_{h\to 0} \frac{-16h - 15h}{h(5x+5h-4)(5x+4)}$$

$$= \lim_{h\to 0} \frac{-31}{(5x+5h-4)(5x+4)}$$

$$= \frac{-31}{(5x+4)^2}$$

(g) $$f(x) = \frac{1}{\sqrt{3x-4}}.$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{3(x+h)-4}} - \frac{1}{\sqrt{3x-4}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x-4} - \sqrt{3x+3h-4}}{h\left(\sqrt{3(x+h)-4}\right)\left(\sqrt{3x-4}\right)}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3x-4} - \sqrt{3x+3h-4}\right)\left(\sqrt{3x-4} + \sqrt{3x+3h-4}\right)}{h\left(\sqrt{3x+3h-4}\right)\left(\sqrt{3x-4}\right)\left(\sqrt{3x-4} + \sqrt{3x+3h-4}\right)}$$

$$= \lim_{h\to 0} \frac{3x-4 - (3x+3h-4)}{h\left(\sqrt{3x+3h-4}\right)\left(\sqrt{3x-4}\right)\left(\sqrt{3x-4} + \sqrt{3x+3h-4}\right)}$$

$$= \lim_{h\to 0} \frac{-3h}{h\sqrt{3x+3h-4}\,\sqrt{3x-4}\left(\sqrt{3x-4} + \sqrt{3x+3h-4}\right)}$$

$$f'(x) = \lim_{h\to 0} \frac{-3}{(\sqrt{3x+3h-4})(\sqrt{3x-4})(\sqrt{3x-4}+\sqrt{3x+3h-4})}$$

$$= \frac{-3}{2(3x-4)\sqrt{3x-4}} = \frac{-3}{2(3x-4)^{3/2}}$$

(h) $f(x) = \dfrac{1}{x^2-4}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2-4} - \frac{1}{x^2-4}}{h}$$

$$= \lim_{h\to 0} \frac{x^2-4-((x+h)^2-4)}{h(x^2-4)((x+h)^2-4)}$$

$$= \lim_{h\to 0} \frac{x^2-4-(x^2+2hx+h^2-4)}{h(x^2-4)[(x+h)^2-4]}$$

$$= \lim_{h\to 0} \frac{-2hx-h^2}{h(x^2-4)[(x+h)^2-4]}$$

$$= \lim_{h\to 0} \frac{-2x-h}{(x^2-4)[(x+h)^2-4]}$$

$$= \frac{-2x}{(x^2-4)^2}$$

2 (i) $f(x) = 4x^7 - \frac{3}{x} + \sqrt{x}$

$f'(x) = 28x^6 + \frac{3}{x^2} + \frac{1}{2\sqrt{x}}$

(ii) $f(x) = 5x^{3/4} - 2x^{5/3} + x^{1/5}$

$f'(x) = 5 \cdot \frac{3}{4} x^{-1/4} - 2 \cdot \frac{5}{3} x^{2/3} + \frac{1}{5} x^{-4/5}$

$= \frac{15}{4} x^{-1/4} - \frac{10}{3} x^{2/3} + \frac{1}{5} x^{-4/5}$

(iii) $f(x) = (x^4 - 2x^3)(x^{2/3} + x + 1)$

$f'(x) = (4x^3 - 6x^2)(x^{2/3} + x + 1) + (x^4 - 2x^3)(\frac{2}{3}x^{-1/3} + 1)$.

(iv) $f(x) = (2x^{3/4} - x^3)(x^{1/2} - 4x^{2/3})(5x^6 - 3x^3 + 7x^2)$

$f'(x) = (2 \cdot \frac{3}{4}x^{-1/4} - 3x^2)(x^{1/2} - 4x^{2/3})(5x^6 - 3x^3 + 7x^2)$

$+ (2x^{3/4} - x^3)(\frac{1}{2}x^{-1/2} - \frac{8}{3}x^{-1/3})(5x^6 - 3x^3 + 7x^2)$

$+ (2x^{3/4} - x^3)(x^{1/2} - 4x^{2/3})(30x^5 - 9x^2 + 14x)$.

(v) $f(x) = \frac{(5x^7 - 4x^3 + 1)}{(2x^{1/2} - x + 1)}$

$f'(x) = \frac{(35x^6 - 12x^2)(2x^{1/2} - x + 1) - (5x^7 - 4x^3 + 1)(x^{-1/2} - 1)}{(2x^{1/2} - x + 1)^2}$

(vi) $$f(x) = \frac{(2x^3 - x^2 + 1)(4x^5 - x^2)}{(2x^6 - x^3 + 1)}$$

$$f'(x) = \frac{(2x^6 - x^3 + 1)\left[(6x^2 - 2x)(4x^5 - x^2)\right] + (2x^3 - x^2 + 1)(4x^5 - x^2)\left[12x^5 - 3x^2\right]}{(2x^6 - x^3 + 1)^2}$$

(vii) $$f(x) = \frac{(x^{1/2} + 1)(x^{2/3} - x^{3/4})}{(x^2 + 1)(3x^3 - 7x^2 + 1)}$$

$$f'(x) = \frac{\left[\frac{1}{2}x^{-1/2}(x^{2/3} - x^{3/4}) + (x^{1/2} + 1)\left(\frac{2}{3}x^{-1/3} - \frac{3}{4}x^{-1/4}\right)\right](x^2 + 1)(3x^3 - 7x^2 + 1) - (x^{1/2} + 1)(x^{2/3} - x^{3/4})\left[2x(3x^3 - 7x^2 + 1) + (x^2 + 1)(9x^2 - 14x)\right]}{(x^2 + 1)^2(3x^3 - 7x^2 + 1)^2}$$

Note: by writing $f(x) = (x^{1/2} + 1)(x^{2/3} - x^{3/4})(x^2 + 1)^{-1}(3x^3 - 7x^2 + 1)^{-1}$

$f'(x)$ can be determined by using the Product rule and the chain rule. Using this approach we obtain the solution below.

~~viii~~ $$\begin{aligned} f'(x) = {} & \tfrac{1}{2}x^{-1/2}(x^{2/3} - x^{3/4})(x^2 + 1)^{-1}(3x^3 - 7x^2 + 1)^{-1} \\ & + (x^{\frac{1}{2}} + 1)\left[\tfrac{2}{3}x^{-1/3} - \tfrac{3}{4}x^{-1/4}\right](x^2 + 1)^{-1}(3x^3 - 7x^2 + 1)^{-1} \\ & + (x^{1/2} + 1)(x^{2/3} - x^{3/4})\left[(-1)(x^2 + 1)^{-2}(2x)\right](3x^3 - 7x^2 + 1)^{-1} \\ & + (x^{1/2} + 1)(x^{2/3} - x^{3/4})(x^2 + 1)^{-1}\left[(-1)(3x^3 - 7x^2 + 1)^{-2}(9x^2 - 14x)\right] \end{aligned}$$