

THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL Handout
MATH 249-01

1. Evaluate the limits:

for a)

Since $\cos\left(\frac{\pi}{2}\right) = 0$ the type is " $\frac{0}{0}$ " so we can use L'Hopital Rule

$$\lim_{x \rightarrow \pi} \frac{\cos\left(\frac{x}{2}\right)}{\pi - x} = \lim_{x \rightarrow \pi} \frac{-\sin\left(\frac{x}{2}\right) \cdot \frac{1}{2}}{-1} = \frac{1}{2} \quad (\sin \frac{\pi}{2} = 1).$$

for b)

The type is " $\frac{DNE}{\infty}$ " but $-1 \leq \cos \frac{x}{2} \leq 1$ and $\pi - x > 0$

$$\text{so} \quad \frac{-1}{\pi - x} \leq \frac{\cos \frac{x}{2}}{\pi - x} \leq \frac{1}{\pi - x}$$

Since both $\lim_{x \rightarrow -\infty} \frac{\pm 1}{\pi - x} = 0$ by Squeeze Theorem $\lim_{x \rightarrow -\infty} \frac{\cos\left(\frac{x}{2}\right)}{\pi - x} = 0$.

for c)

$$\lim_{x \rightarrow 0^+} x \ln x = "0^+(-\infty)" = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = "\frac{-\infty}{+\infty}" \text{ (L'H.)} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0$$

2. Find the domain and the derivative of

$$(a) f(x) = \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)}$$

domain is $D = \{x \neq 0\} = (-\infty, 0) \cup (0, +\infty)$

$$\begin{aligned} \text{by Product and Chain Rules} \quad f'(x) &= \left(\frac{x}{3}\right)' e^{-\sin\left(\frac{3}{x}\right)} + \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \left(-\sin \frac{3}{x}\right)' = \\ &= \frac{1}{3} e^{-\sin\left(\frac{3}{x}\right)} + \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \left(-\cos \frac{3}{x}\right) \left(-\frac{3}{x^2}\right) = e^{-\sin\left(\frac{3}{x}\right)} \left(\frac{1}{3} + \frac{1}{x} \cos \frac{3}{x}\right) \end{aligned}$$

$$\text{OR use log.diff} \quad \ln f = \ln \frac{x}{3} + \ln e^{-\sin \frac{3}{x}} = \ln x - \ln 3 - \sin \frac{3}{x}$$

$$\text{then} \quad \frac{f'}{f} = \frac{1}{x} - 0 - \cos \frac{3}{x} \cdot (3x^{-1})' = \frac{1}{x} - \cos \frac{3}{x} \cdot (-3) x^{-2} = \frac{1}{x} + 3x^{-2} \cos \frac{3}{x}$$

$$\text{so } f'(x) = \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \left[\frac{1}{x} + \frac{3}{x^2} \cos \frac{3}{x}\right] = \dots \text{ as above.}$$

$$(b) f(x) = \frac{\ln(2x - 3)}{e^{-x^2}}$$

for the domain $2x - 3 > 0$ so $x > \frac{3}{2}$ and $D = \left(\frac{3}{2}, +\infty\right)$

we can change the function to a product

$f(x) = e^{x^2} \cdot \ln(2x - 3)$ then by Product and Chain Rules

$$f'(x) = e^{x^2} (x^2)' \ln(2x - 3) + e^{x^2} \cdot \frac{1}{2x - 3} (2x - 3)'$$

$$= e^{x^2} (2x) \ln(2x - 3) + e^{x^2} \cdot \frac{1}{2x-3} \cdot 2 = 2e^{x^2} \left(x \ln(2x - 3) + \frac{1}{2x - 3} \right)$$

3. A

Given $y = \frac{1}{x^2 + x - 2}$ if $y' = -\frac{2x + 1}{(x^2 + x - 2)^2}$ and $y'' = \frac{6x^2 + 6x + 6}{(x^2 + x - 2)^3}$ i.e.

part a)

domain $D = (-\infty, -2) \cup (-2, 1) \cup (1, +\infty)$

since the denominator $x^2 + x - 2 = (x + 2)(x - 1)$ cannot be 0

$x = -2$ and $x = 1$ are V.A.(vertical asymptotes)

since $\lim_{x \rightarrow -2^+} \frac{1}{(x + 2)(x - 1)} = \text{"}\frac{1}{0^+ \cdot (-3)}\text{"} = -\infty$

$\lim_{x \rightarrow -2^-} \frac{1}{(x + 2)(x - 1)} = \text{"}\frac{1}{0^- \cdot (-3)}\text{"} = +\infty$ and $\lim_{x \rightarrow 1^+} \frac{1}{(x + 2)(x - 1)} = \text{"}\frac{1}{3 \cdot 0^+}\text{"} = +\infty$

$\lim_{x \rightarrow 1^-} \frac{1}{(x + 2)(x - 1)} = \text{"}\frac{1}{0^-}\text{"} = -\infty$

No x-intercepts since never $y = 0$, for $x = 0$ $y = -\frac{1}{2}$

For horizontal asymptotes $\lim_{x \rightarrow \pm\infty} \frac{1}{x^2 + x - 2} = \text{"}\frac{1}{\infty}\text{"} = 0$

thus $y = 0$ is H.A. at both ends. We will go back to the range later.

part b)

given $y' = -\frac{2x + 1}{(x^2 + x - 2)^2}$ so obviously $x = -2, 1$ are singular points

and for critical points solve $y' = 0$

so $2x + 1 = 0$ thus $x = -\frac{1}{2}$ is a critical point and $f(-\frac{1}{2}) = [\frac{1}{4} - \frac{1}{2} - 2]^{-1} = -\frac{4}{9}$

testing y' $-^+ -^+ -^+ -_{-2} -^+ -^+ -_{-\frac{1}{2}} -^- -^- -_{-1} -^- -^- -^- -^-$
 y $- -^{incr} - -_{-2} -^{incr} - -_{-\frac{1}{2}} - -^{decr} - -_{-1} - -^{decr} - -$

and the function is **decreasing** on the intervals $(-\frac{1}{2}, 1)$ and $(1, +\infty)$,

it is **increasing** on $(-\infty, -2)$ and on $(-2, -\frac{1}{2})$

local maximum at $x = -\frac{1}{2}$ $y = -\frac{4}{9}$

part c)

given $y'' = \frac{6x^2 + 6x + 6}{(x^2 + x - 2)^3} = \frac{6(x^2 + x + 1)}{(x + 2)^3(x - 1)^3}$ for $x \neq -2, 1$

solve for possible inflection points solve $y'' = 0$

never $x^2 + x + 1 = 0$ since the discriminant $D = 1 - 4 = -3$ is negative

always $x^2 + x + 1 > 0$, only singular points

testing y'' $-^+ -^+ -_2 -^- -^- -^- -^- -_1 -^+ -^+ -$
 y $-_{conc.up} - -_2 -_{conc.down} - - -_1 -_{conc.up} -^+ -$

therefore the function is **concave up** on $(-\infty, -2)$ and on $(1, +\infty)$

and it is **concave down** on $(-2, 1)$. TOGETHER

from the graph we can read the range $R = (-\infty, -\frac{4}{9}] \cup (0, +\infty)$

THERE IS A GAP IN THE RANGE !

3B.

1.step function is a polynomial so $D = (-\infty, +\infty)$

"ends": $\lim_{x \rightarrow \infty} x(4-x)^3 = +\infty \cdot (-\infty) = -\infty$, $\lim_{x \rightarrow -\infty} x(4-x)^3 = -\infty \cdot (+\infty) = -\infty$

No V. or H. asymptotes; when $x = 0, 4$ $y = 0$;

2.step

by Product Rule $y' = 1 \cdot (4-x)^3 + x \cdot 3(4-x)^2 \cdot (-1) = (4-x)^2(4-x-3x) = 4(4-x)^2(1-x)$

solve $y' = 0$ for critical points : $x = 1, 4$

testing y' $-^+ -^+ -^+ -_1 -^- -^- -^- -_4 -^- -^- -^- -$
 y $-_{incr.} - -_1 -_{decr} - -_4 -_{decr} - -$

the function is **incr.** on $(-\infty, 1)$ and **decr** on $(1, +\infty)$ but has horizontal tangents at $x = 1$ and $x = 4$.

also $f(1) = 27$ and $f(4) = 0$, and $x = 1$ is a local maximum

3.step

$y'' = 4 \cdot 2(4-x)(-1)(1-x) + 4(4-x)^2(-1) = -4(4-x)(2-2x+4-x) = -4(4-x)(6-3x) = 12(x-4)(2-x)$

for possible inflection points solve $y'' = 0$ $x = 4, 2$

testing y'' $-^- -^- -^- -_2 -^+ -^+ -_4 -^- -^- -^- -$
 y $-_{conc.down} - -_2 -_{conc.up} - -_4 -_{conc.down} - -$

therefore the function is **concave up** on $(2, 4)$ and concave down on $(-\infty, 2)$ and on $(4, +\infty)$

TOGETHER we can see that the **range** is $(-\infty, 27]$ and 27 is abs. maximum value at $x = 1$.

4. For $f(x) = \frac{1}{\sqrt{2x^2+1}}$

find $f(2) = \frac{1}{\sqrt{9}} = \frac{1}{3}$ and $f'(x) = -\frac{1}{2}(2x^2+1)^{-\frac{3}{2}} \cdot 4x = -2x \cdot \left(\frac{1}{\sqrt{2x^2+1}}\right)^3$,

$f'(2) = -4 \cdot \left(\frac{1}{3}\right)^3 = -\frac{4}{27}$ so the linearization is $L(x) = \frac{1}{3} - \frac{4}{27}(x-2)$ and

$$\frac{1}{\sqrt{2x^2+1}} \doteq \frac{1}{3} - \frac{4}{27}(x-2) \quad \text{for } x \text{ close to } 2.$$

To get $\frac{1}{\sqrt{3}}$ we need $2x^2+1=3$ so $x=\pm 1$ but $x=1$ is closer to 2

so substitute $x=1$: $\frac{1}{\sqrt{3}} \doteq \frac{1}{3} + \frac{4}{27} = \frac{13}{27} = 0.481.$

5. see graphs

6. A

Dimensions: $x \times x \times y$

so the volume $V = x^2y = 18$ given

looking for min of cost $C = 2 \cdot \text{area of the base and lid} + 3 \cdot \text{area of sides} = 2 \cdot 2x^2 + 3 \cdot 4xy$

so $C = 4x^2 + 12xy$

reduce to one variable: $y = \frac{18}{x^2}$ back to the cost $C(x) = 4x^2 + 12x \cdot \frac{18}{x^2} = 4(x^2 + 3 \cdot 18x^{-1})$

for critical points $C'(x) = 4(2x - 3 \cdot 2 \cdot 9x^{-2}) = 8 \cdot \frac{x^3 - 27}{x^2} = 0$ means $x^3 = 27, x = 3$
C.P.

to justify that we have found **minimum**

$C''(x) = 4(2 + 6 \cdot 18x^{-3}) > 0$ for $x > 0$, so the function is conc. up

the solution is $x = 3m, y = \frac{18}{3} = 6m$

B

Draw some rectangles and name the dimensions $x \times y$

given $A = xy = 280$ and we are looking for minimum of the cost

$C = 25(x + 2y) + 10x = 35x + 50y$

reduce to one variable : $y = \frac{280}{x}$ thus $f(x) = 35x + \frac{7 \cdot 40 \cdot 50}{x} = 35 \left(x + \frac{400}{x} \right)$

and $f'(x) = 35 \left(1 - \frac{400}{x^2} \right)$ Solve $f' = 0$ $x^2 = 400$ and $x = 20m$ ($x > 0$)

back to $y = \frac{280}{20} = 14m$

To justify that we have found minimum use $f''(x) = \frac{35 \cdot 800}{x^3} > 0$ for $x > 0$

so f is concave up and the critical point is a minimum.

(OR $f' > 0$ for $x > 20$ and $f' < 0$ for $x < 20$)

Thus the dimensions are 20m \times 14m with one of the longer sides to be fenced.

7. Evaluate:

(a) for $x > 0$

$$\int \frac{3\sqrt{x} - 5}{x\sqrt{x}} dx = \int \frac{3\sqrt{x}}{x\sqrt{x}} dx - \int \frac{5}{x\sqrt{x}} dx = 3 \int \frac{1}{x} dx - 5 \int x^{-\frac{3}{2}} dx =$$
$$= 3 \ln|x| - 5 \cdot (-2)x^{-\frac{1}{2}} + c = 3 \ln x + \frac{10}{\sqrt{x}} + c$$

(b) $\int 2x^3 \sqrt{2x^2 + 3} dx$

by substitution $u = 2x^2 + 3$ $du = 4x dx$ $\frac{1}{2} du = 2x dx$

re-arrange the integral = $\int x^2 \sqrt{2x^2 + 3} \cdot 2x dx = \frac{1}{2} \int (?) \sqrt{u} du = .$

from the substitution $\frac{u - 3}{2} = x^2$ so

the integral = $\frac{1}{2} \int \frac{u - 3}{2} \sqrt{u} du = \frac{1}{4} \int (u - 3) \sqrt{u} du = \frac{1}{4} \int u^{\frac{3}{2}} du - \frac{3}{4} \int u^{\frac{1}{2}} du =$

using $\int x^r dx = \frac{x^{r+1}}{r+1} + c$ for $r \neq -1$

$$= \frac{1}{4} \cdot \frac{2}{5} u^{\frac{5}{2}} - \frac{3}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} + c = (\text{back to } x) = \frac{1}{10} (2x^2 + 3)^{\frac{5}{2}} - \frac{1}{2} (2x^2 + 3)^{\frac{3}{2}} + c \quad \text{for any } x.$$

8. (a) $\int_{\frac{1}{2}}^1 \frac{3^{\frac{1}{x}}}{x^2} dx$

by substitution $u = \frac{1}{x}$ $du = -\frac{1}{x^2} dx$ $-du = \frac{1}{x^2} dx$ and

x	u
$\frac{1}{2}$	2
1	1

the integral = $-\int_2^1 3^u du = \left[\frac{3^u}{\ln 3} \right]_1^2 = \frac{1}{\ln 3} [3^2 - 3] = \frac{6}{\ln 3}$

using $\int a^x dx = \frac{1}{\ln a} a^x$ for $a > 0, a \neq 1$

(b) $\int_0^1 \frac{4x + 3}{3 - 2x} dx$

by substitution $u = 3 - 2x$ $du = -2dx$ $-\frac{1}{2} du = dx$ and

x	u
0	3
1	1

the integral = $-\frac{1}{2} \int_3^1 \frac{?}{u} du = \frac{1}{2} \int_1^3 \frac{?}{u} du =$ from the substitution

$2x = 3 - u$ so $4x = 6 - 2u$ and $4x + 3 = 9 - 2u$

therefore the integral = $\frac{1}{2} \int_1^3 \frac{9 - 2u}{u} du = \frac{9}{2} \int_1^3 \frac{1}{u} du - \int_1^3 du = \frac{9}{2} [\ln|u|]_1^3 - [3 - 1] =$

$$= \frac{9}{2} \ln 3 - 2$$

9. a)

simplify first $\ln(x + 1)^2 = \ln(4x)$ then apply exp. f.

$$(x + 1)^2 = 4x \quad x^2 - 2x + 1 = 4x$$

$$x^2 - 2x + 1 = 0 \quad (x - 1)^2 = 0$$

$x = 1$ is the solution.

b)

cross multiply first $4^{3x} = 5 \cdot 2^{2x+1}$, then apply \ln to both sides

thus

$$3x \ln 4 = \ln 5 + (2x + 1) \ln 2 \quad 3x \ln 4 - 2x \ln 2 = \ln 5 + \ln 2 = \ln(5 \cdot 2)$$

$$3x \ln 2^2 - 2x \ln 2 = (6x - 2x) \ln 2 = 4x \ln 2 = \ln 10$$

$$\text{or} \quad x(3 \ln 4 - 2 \ln 2) = x \ln \frac{4^3}{2^2} = x \ln 16$$

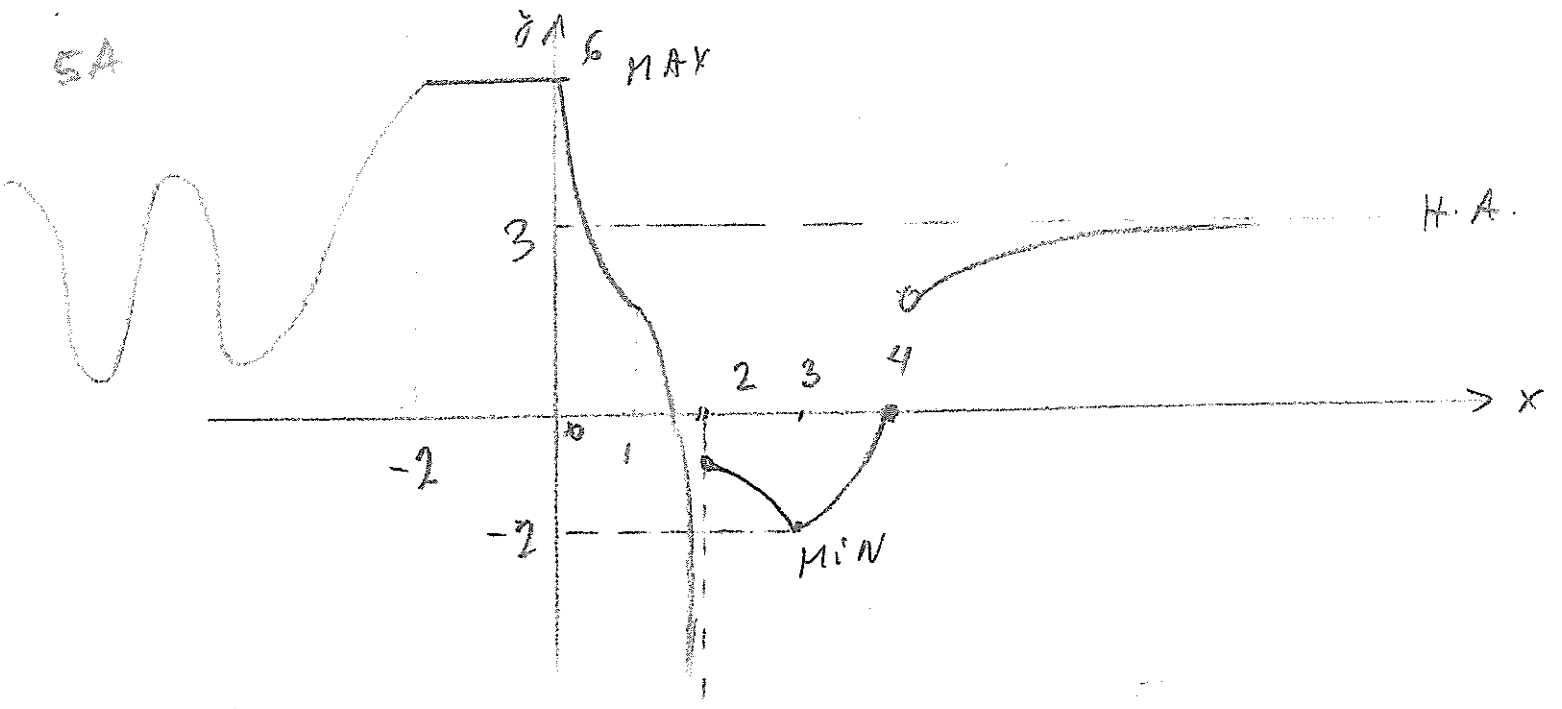
$$\text{and finally} \quad x \ln 2^4 = \ln 10 \quad x = \frac{\ln 10}{\ln 16} = \frac{\ln 10}{4 \ln 2}$$

OR

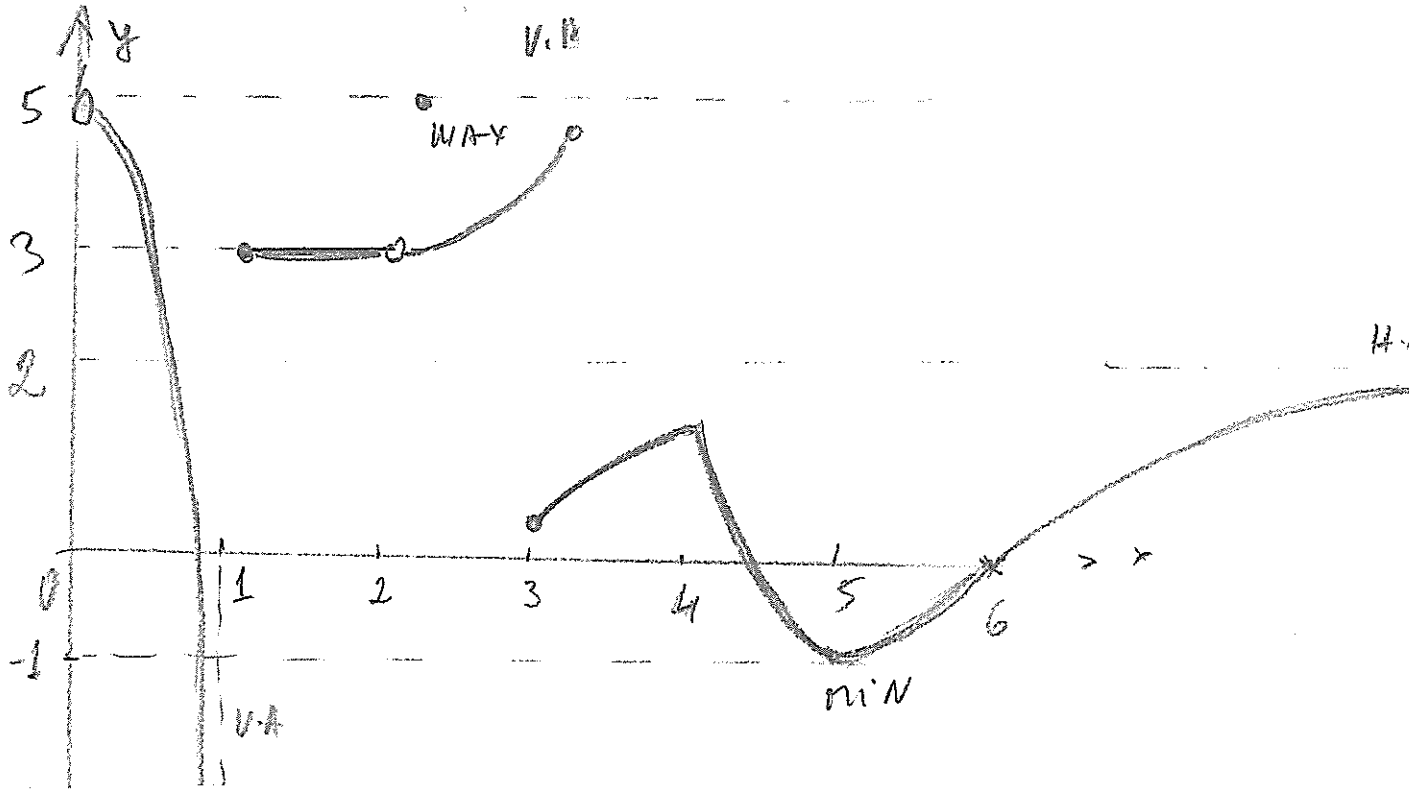
$$\frac{4^{3x}}{2^{2x+1}} = \frac{2^{6x}}{2^{2x+1}} = 2^{4x-1} = 5 \quad 2^{4x} = 10$$

$$\text{and then apply } \log_2 \text{ to both sides:} \quad 4x = \log_2 10 \quad x = \frac{1}{4} \log_2 10.$$

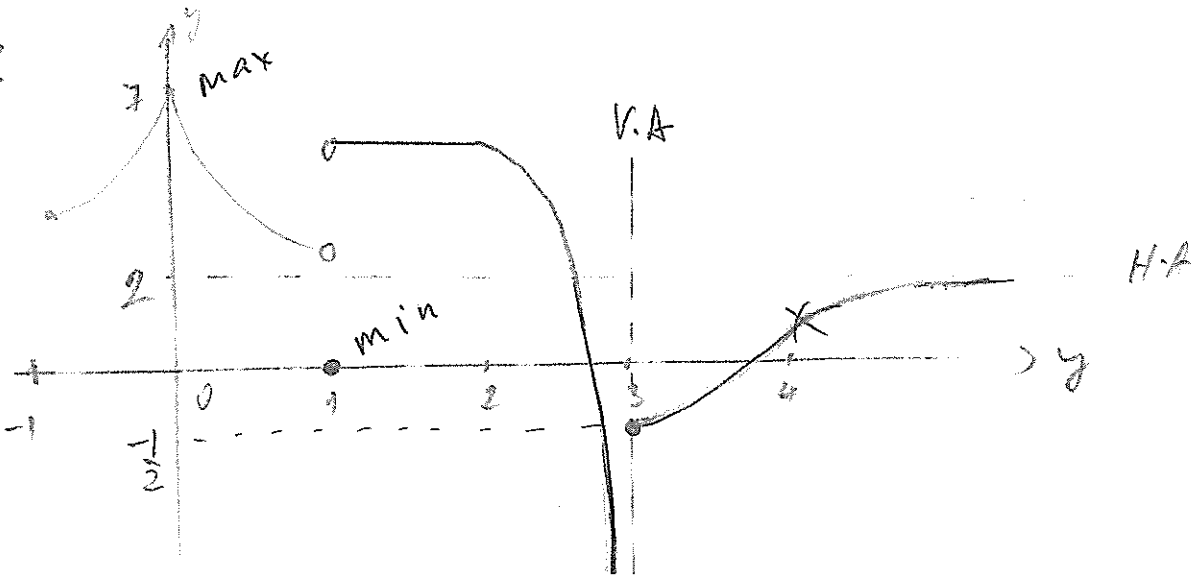
5A



5B

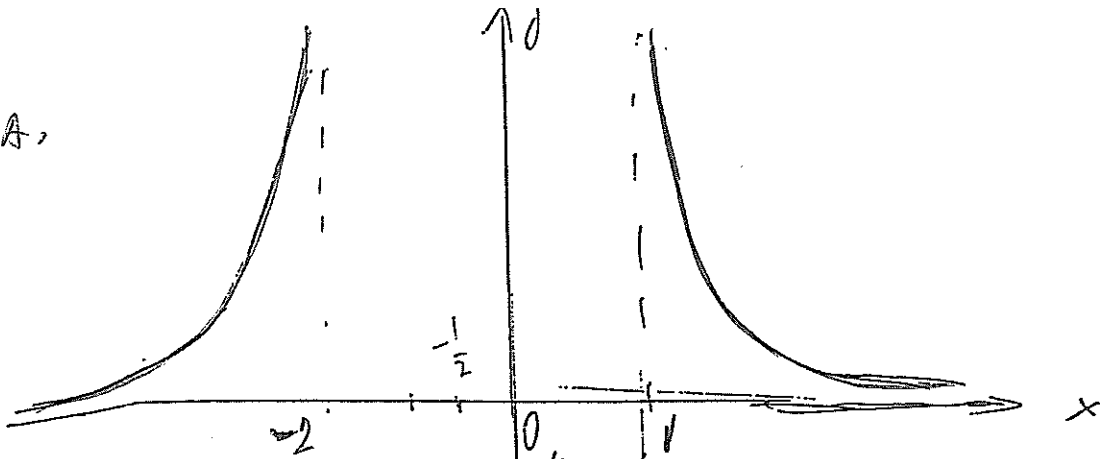


5C



3A.

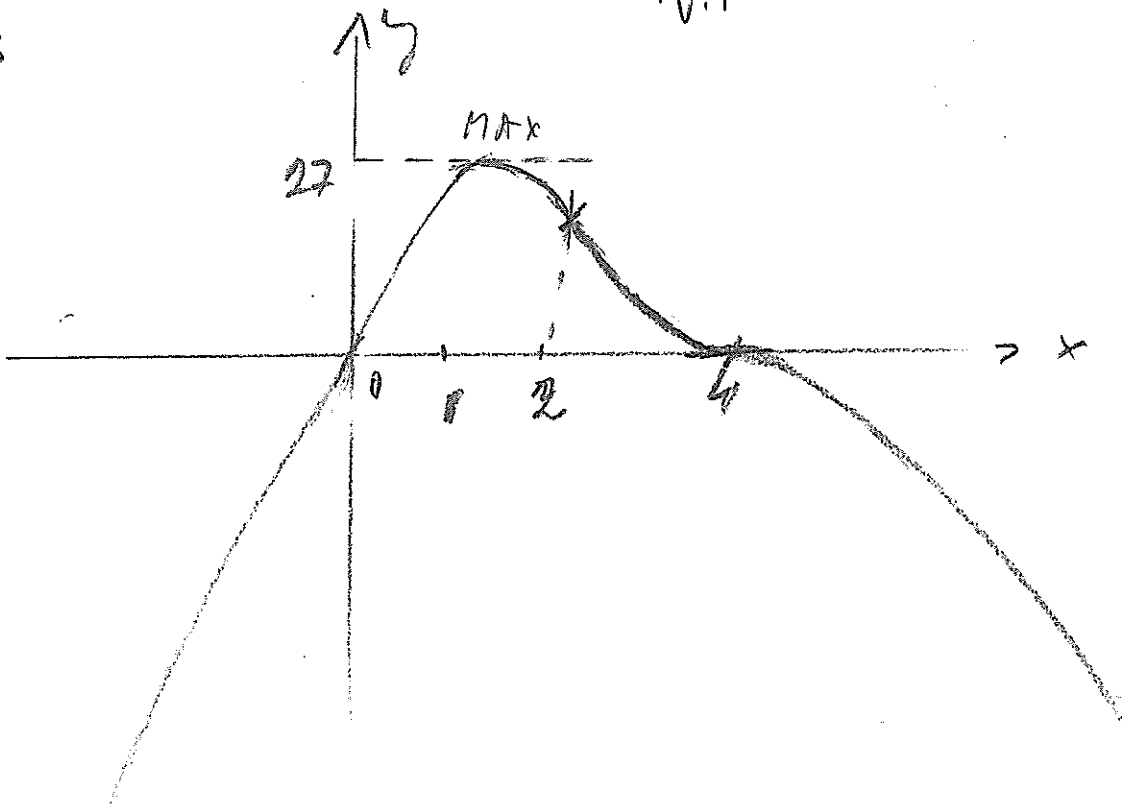
H.A.



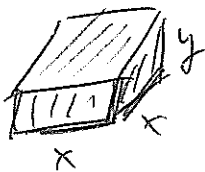
V.A

V.A

3B



6.A



6.B.

