THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 249 (L01, L03)

Fall 2002 Time: 2 Hours
December 19, 2002

NO GRAPHING CALCULATORS
Evalmuate the limits:

1. (a) $\lim_{x\to+\infty} \frac{\ln x}{\sqrt[4]{x}} = \text{"}\frac{\infty}{\infty}\text{"}(\text{L'H.R.}) = \lim_{x\to+\infty} \frac{x^{-1}}{\frac{1}{4}x^{-\frac{3}{4}}} = \lim_{x\to+\infty} 4x^{-\frac{1}{4}} = \text{"}\frac{4}{\infty}\text{"} = 0$

   (b) $\lim_{x\to-\infty} \frac{x^3}{e^{2x}} = \text{"}\frac{-\infty}{0}\text{"}(\text{NOT L'H.R.}) = \lim_{x\to-\infty} x^3 e^{-2x} = (-\infty)(+\infty) = -\infty$
   since $\text{"}e^{-\infty}\text{"} = 0$ and $\text{"}e^{+\infty}\text{"} = +\infty$

   (c) $\lim_{x\to+\infty} \frac{x^3}{e^{2x}} = \text{"}\frac{\infty}{\infty}\text{"}(\text{L'H.R.}) = \lim_{x\to+\infty} \frac{3x^2}{2e^{2x}} = (\text{again}) = \lim_{x\to+\infty} \frac{6x}{4e^{2x}} =$
   $(\text{again}) = \lim_{x\to+\infty} \frac{6}{8e^{2x}} = 0$

   (d) $\lim_{x\to-\infty} \frac{\sin 3x}{x^2} = 0$
   by Squ.Th. since $-1 \le \sin 3x \le 1$, $\frac{-1}{x^2} \le \frac{\sin 3x}{x^2} \le \frac{1}{x^2}$
   and $\lim_{x\to+\infty} \frac{\pm 1}{x^2} = 0.$

2. Find the domain and the derivative of f of

   (a) $f(x) = \frac{\sqrt{3x}}{e^{\cos x}}$ (you may use log.differentiation)
   $\ln f(x) = \ln(3x)^{\frac{1}{2}} - \ln e^{\cos x} = \frac{1}{2}(\ln 3 + \ln x) - \cos x$
   $\frac{f'(x)}{f(x)} = \frac{1}{2x} + \sin x$ and $f'(x) = \frac{\sqrt{3x}}{e^{\cos x}}\left(\frac{1}{2x} + \sin x\right)$
   also by Product Rule
   $f'(x) = \frac{1}{2}(3x)^{-\frac{1}{2}} \cdot 3e^{-\cos x} + \sqrt{3x}e^{-\cos x}\sin x$ for $x > 0$;

   (b) $g(x) = \frac{\ln(2x+5)}{2+x}$ *by* Quotient Rule
   $g'(x) = \frac{\frac{2}{2x+5}(x+2) - \ln(2x+5)}{(x+2)^2}$ for $2x+5 > 0$ and $x \neq -2$
   so $x \in \left(\frac{-5}{2}, -2\right) \cup (-2, \infty)$

(c) $h(x) = \sin(4x^2 + \sqrt{x})$ by Chan Rule $h'(x) = \cos\left(4x^2 + \sqrt{x}\right)\left(8x + \dfrac{1}{2\sqrt{x}}\right)$ for $x > 0$.

3. Sketch the graph of $y = 1 - \dfrac{6}{x} + \dfrac{8}{x^3}$ i.e.

   (a) find the domain, range, vertical and horizontal asymptotes,

   (b) find the intervals where f is increasing or decreasing;

   (c) find the intervals where f is concave down or up

   USE $y' = \dfrac{6x^2 - 24}{x^4}$ and $y'' = \dfrac{12(8 - x^2)}{x^5}$

domain $D = \{x \neq 0\}$ and $x = 0$ is a V.A.; $\lim\limits_{x\to\pm\infty}\left(1 - \dfrac{6}{x} + \dfrac{8}{x^3}\right) = 1$ so $y = 1$ is a H.A.

$\lim\limits_{x\to 0^\pm} \dfrac{1}{x^3}\left(x^3 - 6x^2 + 8\right) = \pm\infty$ the range is $R = (-\infty, +\infty)$

$y' = \dfrac{6x^2 - 24}{x^4} = \dfrac{6(x^2 - 4)}{x^4}$ does not exist for $x = 0$....sing.point

and $y' = 0$ for $x = \pm 2$ ...critical points

testing y' $- - -^{pos} - -_{-2} - -^{neg} - -_{0} -^{neg} - -_{2} -^{pos} - - -$

so By First Der.Test function is incr. on $(-\infty, -2)$ and on $(2, \infty)$

decr. on $(-2, 0)$ and on $(0, 2)$

values $f(2) = -1$... loc.min and $f(-2) = 3$...loc max

$y'' = \dfrac{12(8 - x^2)}{x^5}$ does not exist for $x = 0$ ....sing.point

and $y'' = 0$ for $x = \pm\sqrt{8}$

testing y'' $- - -^{pos} - -_{-\sqrt{8}} - -^{neg} - -_{0} -^{pos} - - \sqrt{8} -^{neg} - - -$

so By Second Der.Test function is concave up . on $\left(-\infty, -\sqrt{8}\right)$ and on $\left(0, \sqrt{8}\right)$

concave down on $\left(-\sqrt{8}, 0\right)$ and on $\left(\sqrt{8}, \infty\right)$.

4. (a) Find the tangent approximation (linearization) of

$$f(x) = \sqrt[3]{25 + 2x} \text{ around } x_0 = 1.$$

   (b) Use it to estimate $\sqrt[3]{26}$.

for a)

$f(1) = \sqrt[3]{27} = 3$ $f'(x) = \dfrac{1}{3}(25 + 2x)^{-\frac{2}{3}} \cdot 2$ $f'(1) = \dfrac{2}{3\,(3)^2} = \dfrac{2}{27}$ and

the linearization is $L(x) = 3 + \dfrac{2}{27}(x - 1)$

for b)

$$f(x) \doteq L(x) \text{ for } x = \frac{1}{2} \text{ we get} \qquad \sqrt[3]{26} \doteq 3 + \frac{2}{27}\left(-\frac{1}{2}\right) = 3 - \frac{1}{27} = \frac{81}{27}.$$

5. Sketch a graph of <u>one</u> function f satisfying all the following conditions:

   (a) f is defined on $(-\infty, +\infty)$, continuous there except

   (b) f is discontinuous at $x = -1, -3$ where $\lim_{x \to -3^-} f(x) = f(-3) = -2$, $\quad x = -1$ is a vertical asymptote.

   (c) $y = 2$ is a horizontal asymptote and $\lim_{x \to +\infty} f(x)$ <u>does not exist</u>,

   (d) f is increasing on $(-1, 1)$ and on $(-2, -1)$, f is decreasing on $(-3, -2)$ and on $(-\infty, -3)$, and $f'(x) = 0$ for all $x \in (1, 3)$;

   (e) f is concave up on $(0, 1)$ and on $(-3, -2)$; f is concave down on $(-1, 0)$, on $(-2, -1)$ and on $(-\infty, -3)$;

   (f) absolute maximum value is 8, and local minimum value is -1.

6. A company plans to manufacture canned goods and wants to minimize the cost of material needed to make a closed cylindrical can

   holding $500\pi \text{cm}^3$.Find the radius and height of the can if the material for the top and bottom costs 6 cents per cm^2 and the material for the side

   costs 3 cents per cm^2.

the volume $V = \pi r^2 h = 500\pi$ so $\qquad h = \dfrac{500}{r^2}$

now the cost $\qquad C = 2\pi r^2 \cdot 6 + 2\pi r h \cdot 3$ and $C(r) = 6\pi\left(2r^2 + \dfrac{500}{r}\right)$

looking for min of cost

so $\quad C'(r) = 6\pi\left(4r - \dfrac{500}{r^2}\right) = 6\pi\left(\dfrac{4r^3 - 500}{r^2}\right) = 0 \qquad$ for $r^3 = 125$

critical point $\qquad r = 5$ cm and then $\qquad h = \dfrac{500}{25} = 20$ cm

now, justification that we have found the minimum:

since $C''(r) = 6\pi\left(4 + \dfrac{1000}{r^3}\right) > 0$ for $r > 0$, C is concave up and we got min.

7. Find

   (a) since it must $\ln x \geq 0 \qquad$ for $x \geq 1 \qquad$ by subst. $\ln x = u$, $\dfrac{dx}{x} = du$

   $$\int \frac{\sqrt{\ln x}}{x} dx = \int \sqrt{u}\, du = \frac{2}{3}(u)^{\frac{3}{2}} + c = \frac{2}{3}(\ln x)^{\frac{3}{2}} + c$$

(b) for $x > 0$

$$\int \left(\frac{2}{\sqrt{x}} - \frac{\sqrt{x}}{2}\right)^2 dx = \int \left(\frac{4}{x} - 2\frac{2}{\sqrt{x}} \cdot \frac{\sqrt{x}}{2} + \frac{x}{4}\right) dx = 4\ln x - 2x + \frac{x^2}{8} + c$$

(c) for any x

$$\int x \sin(3x^2)\, dx = (\text{by subst. } u = 3x^2, du = 6xdx) = \frac{1}{6}\int \sin u\ du =$$

$$= \frac{-\cos u}{6} + c = -\frac{1}{6}\cos(3x^2) + c.$$

8. Evaluate

(a) $\displaystyle\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}} dx == \int_1^2 2^u du = \left[\frac{2^u}{\ln 2}\right]_1^2 = \frac{3}{\ln 2}$

by subst. $u = \sqrt{x}, du = \dfrac{dx}{2\sqrt{x}},$ $\begin{array}{cc} x & u \\ 1 & 1 \\ 4 & 2 \end{array}$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} \sin x\ \cos^4 x\ dx == -\int_1^{\frac{1}{2}} u^4\ du = \left[\frac{u^5}{5}\right]_{\frac{1}{2}}^1 = \frac{1}{4}\left[1 - \frac{1}{2^5}\right] = \frac{31}{128}$

by subst. $u = \cos x, du = -\sin x\ dx,$ $\begin{array}{cc} x & u \\ 0 & 1 \\ \frac{\pi}{3} & \frac{1}{2} \end{array}$

(c) $\displaystyle\int_{-1}^{0} (\sqrt[3]{x} + 5x^4) dx = \left[\frac{3}{4}x^{\frac{4}{3}}\right]_{-1}^0 + [x^5]_{-1}^0 = 0 - \frac{3}{4} + 0 - (-1)^5 = \frac{1}{4}.$

END OF EXAMINATION

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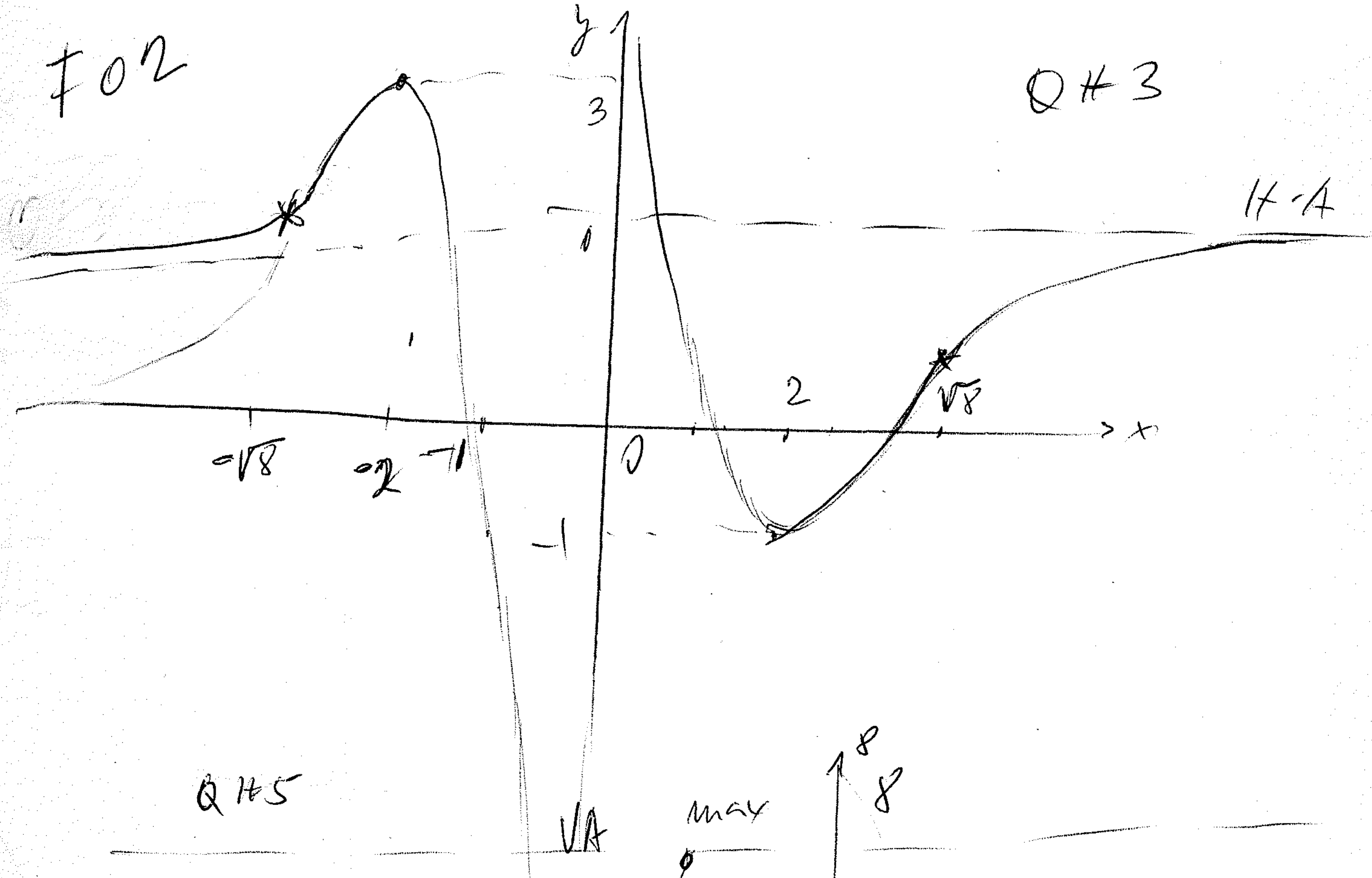
Q#3



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