jThe University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #4

1. Find an equation of the tangent line to

$$\sqrt{x^2 - y} = \frac{9x}{y} - 1$$

at the point P $(5, 9)$.

2. Find a general antiderivative of $f(x) = \dfrac{5\sqrt{x} - 6x^3 - 8x^2 + 3}{x^2}$ for $x > 0$.

3. Solve $y'' = 2\sin(\pi - 2x)$ with $y'(\pi) = 0$ and $y(\pi) = 3$.

4. Find the second derivative of $f(x) = x\cos(x^2)$.Simplify.

5. Find y' in terms of x and y if $\quad 2x + 3y = \dfrac{y^2}{x+1}$.

6. Find a general antiderivative of $\quad f(x) = \dfrac{1}{\cos^2(3x - 1)}$ in the domain (find the domain).

7. Solve $y'' = \dfrac{3}{\sqrt{x}} - 6x$, $y'(4) = 2, y(4) = 0$.

8. Find the second derivative of $y = \dfrac{1}{1 + x^2}$. Simplify.

9. Find an equation of the tangent line at the point $(6, \pi)$ to

$$2\cos\frac{y}{x} + \frac{xy}{6} = \sqrt{3} + \pi.$$

10. Solve (i.e. find y including an interval)

$$y' = \frac{1}{(5 - x)^3}$$

with $\qquad y(4) = 1$

11. Find $\qquad \displaystyle\int \left(3\sqrt{x} - \frac{1}{3x}\right)^2 dx$ for $x > 0$.

SOLUTIONS

For 1)

Use implicit differentiation and Chain Rule on the left ,Quotient Rule on right:

$\frac{1}{2}\left(x^2-y\right)^{-\frac{1}{2}}\cdot\left(x^2-y\right)'=9\cdot\left(\frac{x}{y}\right)'-0$

$\frac{1}{2}\left(x^2-y\right)^{-\frac{1}{2}}\cdot\left(2x-y'\right)=9\cdot\frac{1\cdot y-x\cdot y'}{y^2}$

now,$x=5,y=9,y'=m$

$\frac{1}{2}\left(25-9\right)^{-\frac{1}{2}}\left(10-m\right)=9\cdot\frac{9-5m}{9^2}$ so $\frac{1}{8}\left(10-m\right)=\frac{1}{9}\left(9-5m\right)$

multiply by $9\cdot 8$

$90-9m=72-40m$ thus $31m=-18$ and

$m=-\frac{18}{31}$ and an equation is $y=-\frac{18}{31}\left(x-5\right)+9.$

For 2)

get rid of the quotient

$\int f(x)dx=5\int\frac{\sqrt{x}}{x^2}dx-6\int\frac{x^3}{x^2}dx-8\int\frac{x^2}{x^2}dx+3\int\frac{1}{x^2}dx=$

$5\int x^{-\frac{3}{2}}dx-6\int xdx-8\int dx+3\int x^{-2}dx+c=5\left(-2\right)x^{-\frac{1}{2}}-6\cdot\frac{x^2}{2}-8x+3\cdot\frac{x^{-1}}{-1}+c$

$=-\frac{10}{\sqrt{x}}-3x^2-8x-\frac{3}{x}+c$ for $x>0$.

For 3)

$y'=\int y''dx=2\int\sin\left(\pi-2x\right)dx=2\cdot\frac{-\cos\left(\pi-2x\right)}{-2}+c_1=\cos\left(\pi-2x\right)+c_1$

using $\int\sin(ax+b)dx=\frac{-\cos(ax+b)}{a}+c_1$ $a=-2,b=\pi$

now use the condition $y'=0$ for $x=\pi$

$0=\cos\left(-\pi\right)+c_1=-1+c_1$ so $c_1=1$ and $y'=\cos\left(\pi-2x\right)+1$

I*ntegrate again using $\int\cos(ax+b)dx=\frac{\sin(ax+b)}{a}+c$

$y=\int y'dx=\int\cos\left(\pi-2x\right)dx+\int 1dx+c_2=\frac{\sin\left(\pi-2x\right)}{-2}+x+c_2=-\frac{1}{2}\sin\left(\pi-2x\right)+x+c_2$

use the second condition $y=3$ for $x=\pi$

$3=-\frac{1}{2}\sin\left(-\pi\right)+\pi+c_2=0+\pi+c_2$ so $c_2=3-\pi$

and the solution is

$$y=-\frac{1}{2}\sin\left(\pi-2x\right)+x+3-\pi$$

For 4)

by Product and Chain Rules

$f'(x)=\left[x\cos(x^2)\right]'=(x)'\cdot\cos\left(x^2\right)+x\left(-\sin x^2\right)\left(x^2\right)'=\cos\left(x^2\right)-2x^2\sin(x^2)$

again

$f''(x)=\left[\cos\left(x^2\right)-2x^2\sin(x^2)\right]'=-2x\sin(x^2)-4x\sin(x^2)-2x^2\cos(x^2)2x=$

$=-6x\sin(x^2)-4x^3\cos(x^2)$

For 5)

use implicit differentiation,Quotient and Chain Rules:

$$(2x+3y)'=\left(\frac{y^2}{x+1}\right)'$$

$2+3y' = \dfrac{2yy'\,(x+1) - y^2}{(x+1)^2}$ multiply both side by $(x+1)^2$

$2\,(x+1)^2 + 3y'\,(x+1)^2 = 2yy'(x+1) - y^2$ all terms with y'

$y'\left[3\,(x+1)^2 - 2y\,(x+1)\right] = -y^2 - 2(x+1)^2$

so

$$y' = \frac{-y^2 - 2(x+1)^2}{3\,(x+1)^2 - 2y\,(x+1)} \text{ if the denominator is not 0.}$$

OR

we can simplify first by multiplying the original expression by $(x+1)$

$$2x + 3y = \frac{y^2}{x+1} \qquad (2x+3y)\,(x+1) = y^2$$

then

$2x^2 + 2x + 3xy + 3y = y^2$ then differentiate by Pr.and Chain rules:

$4x + 2 + 3y + 3xy' + 3y' = 2yy' \qquad 4x + 2 + 3y = y'(2y - 3x - 3)$

then

$$y' = \frac{4x+2+3y}{2y-3x-3} \text{ if the denominator is not 0.}$$

Notice that it looks different because we have a relation between x and y .

For 6)

$\displaystyle\int \frac{1}{\cos^2(3x-1)}dx = \tfrac{1}{3}\tan(3x-1) + c$

since $(\tan 3x - 1)' = \sec^2(3x-1)\cdot 3 = \dfrac{3}{\cos^2(3x-1)}$

for $\quad 3x - 1 \neq \frac{\pi}{2} + k\pi$ so $x \neq \frac{1}{3} + \frac{\pi}{6} + k\frac{\pi}{3} \qquad k = 0, \pm1, \pm2, \pm3, ...$

For 7)

$y'' = \dfrac{3}{\sqrt{x}} - 6x \qquad y'(4) = 2 \qquad y(4) = 0$ for $x > 0$

$y' = \displaystyle\int y'' dx = 3\int x^{-\frac{1}{2}}dx - 6\int x dx + c_1 = 6\sqrt{x} - 3x^2 + c_1$

now $x = 4 \qquad y' = 2 \qquad$ solve for c_1 :

$2 = 6\cdot 2 - 3\cdot 16 + c_1 \qquad c_1 = 38$

so

$$y' = 6\sqrt{x} - 3x^2 + 38 \qquad \text{for } x > 0$$

integrate again

$y = \displaystyle\int y' dx = 6\int x^{\frac{1}{2}}dx - 3\int x^2 dx + 38\int dx = 6\cdot\tfrac{2}{3}x^{\frac{3}{2}} - 3\cdot\tfrac{x^3}{3} + 38x + c_2$

$y = 4x^{\frac{3}{2}} - x^3 + 38x + c_2$

now $x = 4 \qquad y = 0 \qquad$ solve for c_2 :

$0 = 4\cdot 2^3 - 4^3 + 38\cdot 4 + c_2 \qquad c_2 = -4(8 - 16 + 38) = -120$

thus the solution of the given problem is

$$y = 4x^{\frac{3}{2}} - x^3 + 38x - 120 \qquad \text{for any } x > 0.$$

For 8)

by Chain Rule

$y' = \left(\dfrac{1}{1+x^2}\right)' = \left[(1+x^2)^{-1}\right]' = (-1)\,(1+x^2)^{-2}\,2x = -2x(1+x^2)^{-2}$

by product and chain rules

$y'' = (-2x)'\,(1+x^2)^{-2} - 2x\,[(1+x^2)^{-2}]' = -2(1+x^2)^{-2} - 2x\,(-2)\,(1+x^2)^{-3}2x =$

$= -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$
Or by Quotient Rule
$y' = \left(\dfrac{1}{1+x^2}\right)' = \dfrac{0-2x}{(1+x^2)^2} \qquad y'' = \left(\dfrac{-2x}{(1+x^2)^2}\right)' =$
$\dfrac{-2(1+x^2)^2 + 2x2\,(1+x^2)\,2x}{(1+x^2)^4} = \dfrac{(1+x^2)\left[-2(1+x^2)+8x^2\right]}{(1+x^2)^4} = \dfrac{-2+6x^2}{(1+x^2)^3}$

For 9)

Use implicit differentiation: $\quad 2\left[\cos\frac{y}{x}\right]' + \frac{1}{6}(xy)' = \left(\sqrt{3}+\pi\right)'$
by Chain , Quotient and Product Rules:
$2\left(-\sin\frac{y}{x}\right)\left(\dfrac{y}{x}\right)' + \frac{1}{6}(1\cdot y + x\cdot y') = 0$
$-2\sin\frac{y}{x}\cdot\dfrac{y'\cdot x - y\cdot 1}{x^2} + \frac{1}{6}(y+xy') = 0$
Now $x=6, y=\pi$,and $y'=m$:
$-2\sin\frac{\pi}{6}\cdot\dfrac{6m-\pi}{36} + \frac{1}{6}(\pi+6m) = 0$, multiply both sides by 36 and use $\sin\frac{\pi}{6} = \frac{1}{2}$
thus
$-(6m-\pi) + 6(\pi+6m) = 0$ and the equation is now:$-6m+\pi+6\pi+36m = 0$,
thus
$30m = -7\pi$ and $m = -\dfrac{7\pi}{30}$.The equation of the tangent line is :

$$y = -\frac{7\pi}{30}(x-6) + \pi$$

.

For 10)

For $x \neq 5 \qquad y = \int y'dx = \int (5-x)^{-3}\,dx = \dfrac{(5-x)^{-2}}{-2(-1)} + c = \dfrac{1}{2(5-x)^2} + c$
using $\int (ax+b)^r\,dx = \dfrac{(ax+b)^{r+1}}{a(r+1)} + c$,where $a=-1, b=5, r=-3$
now if $x=4, y=1$ solve for c: $\qquad 1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$.
Together the solution is $y = \frac{1}{2}(5-x)^{-2} + \frac{1}{2}$ for $x \in (-\infty, 5)$
since the condition is at $x = 4 < 5$.

For 11)

$\int \left(3\sqrt{x} - \dfrac{1}{3x}\right)^2 dx$
(get rid of the power using $(A-B)^2 = A^2 - 2AB + B^2$)
$= \int \left[(3\sqrt{x})^2 - 2\cdot 3\sqrt{x}\cdot\dfrac{1}{3x} + \left(\dfrac{1}{3x}\right)^2\right] dx =$
$= 9\int xdx - 2\int x^{-\frac{1}{2}}dx + \frac{1}{9}\int x^{-2}dx = 9\cdot\frac{1}{2}x^2 - 2\cdot 2x^{\frac{1}{2}} + \frac{1}{9}\cdot\dfrac{x^{-1}}{-1} + c$
$y = \dfrac{9}{2}x^2 - 4\sqrt{x} - \dfrac{1}{9x} + c \qquad$ for $x > 0$.