MATHEMATICS 249 L04 MIDTERM SOLUTIONS November 9, 2010

[5] 1. Find $\lim_{x\to 1}\left(\dfrac{2-\sqrt{3x+1}}{2-2x}\right)$. Do not use l'Hôpital's Rule.

Solution:

$$\begin{aligned}\lim_{x\to 1}\left(\frac{2-\sqrt{3x+1}}{2-2x}\right) &= \lim_{x\to 1}\left(\frac{(2-\sqrt{3x+1})(2+\sqrt{3x+1})}{(2-2x)(2+\sqrt{3x+1})}\right)\\ &= \lim_{x\to 1}\left(\frac{4-(3x+1)}{(2-2x)(2+\sqrt{3x+1})}\right)\\ &= \lim_{x\to 1}\left(\frac{3-3x}{(2-2x)(2+\sqrt{3x+1})}\right)\\ &= \lim_{x\to 1}\left(\frac{3(1-x)}{2(1-x)(2+\sqrt{3x+1})}\right)\\ &= \lim_{x\to 1}\left(\frac{3}{2(2+\sqrt{3x+1})}\right)\\ &= \frac{3}{2(2+\sqrt{3+1})}=\frac{3}{2(2+2)}=\frac{3}{8}\ .\end{aligned}$$

[5] 2. Find $\lim_{x\to 5^+}\left(\dfrac{x^2+5}{5-x}\right)$. If your answer can be written as ∞ or $-\infty$, do so.

Solution. If x is close to 5 but bigger than 5, x^2+5 is close to $25+5=30$, and $5-x$ is close to 0 and **less** than 0. Therefore $\dfrac{x^2+5}{5-x}$ is large and **negative**. Therefore

$$\lim_{x\to 5^+}\left(\frac{x^2+5}{5-x}\right)=-\infty.$$

[5] 3. Find $\dfrac{d}{dx}(e^{2x}\sec^3 x)$.

Solution. By the Product Rule and the Chain Rule,

$$\frac{d}{dx}(e^{2x}\sec^3 x)=(e^{2x}\cdot 2)\sec^3 x+e^{2x}(3\sec^2 x\cdot\sec x\tan x)=e^{2x}\sec^3 x\cdot(2+3\tan x).$$

[5] 4. Use implicit differentiation to find y' where $x^2y-\ln x=4\sin x+\cos y$.

Solution. We get

$$\frac{d}{dx}\left(x^2y-\ln x\right)=\frac{d}{dx}\left(4\sin x+\cos y\right)$$

which by the Product Rule and the Chain Rule becomes

$$2xy + x^2y' - \frac{1}{x} = 4\cos x - \sin y \cdot y'.$$

Rearranging, we get

$$x^2y' + \sin y \cdot y' = 4\cos x - 2xy + \frac{1}{x} \ ,$$

so

$$y'(x^2 + \sin y) = 4\cos x - 2xy + \frac{1}{x} \ ,$$

and so

$$y' = \frac{4\cos x - 2xy + 1/x}{x^2 + \sin y} \ .$$

[5] 5. Find $\dfrac{d}{dx}\left(\dfrac{x^3}{7-\sqrt{x}}\right)$.

Solution. By the Quotient Rule,

$$\frac{d}{dx}\left(\frac{x^3}{7-\sqrt{x}}\right) = \frac{(7-\sqrt{x}) \cdot 3x^2 - x^3\left(-\dfrac{1}{2\sqrt{x}}\right)}{(7-\sqrt{x})^2} \ .$$

[5] 6. USE THE LIMIT DEFINITION OF DERIVATIVE to find $\dfrac{d}{dx}\left(-\dfrac{5}{x}\right)$.

Solution. We get

$$\begin{aligned}
\frac{d}{dx}\left(-\frac{5}{x}\right) &= \lim_{h\to 0}\left(\frac{-\dfrac{5}{x+h} - \left(-\dfrac{5}{x}\right)}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{-\dfrac{5}{x+h} + \dfrac{5}{x}}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{1}{h} \cdot \frac{-5x + 5(x+h)}{x(x+h)}\right) \\
&= \lim_{h\to 0}\left(\frac{-5x + 5x + 5h}{hx(x+h)}\right) \\
&= \lim_{h\to 0}\left(\frac{5h}{hx(x+h)}\right) \\
&= \lim_{h\to 0}\left(\frac{5}{x(x+h)}\right) = \frac{5}{x^2} \ .
\end{aligned}$$

[5] 7. Use the derivatives of $\sin x$ and $\cos x$ to find the formula for $\dfrac{d}{dx}(\tan x)$.

Solution. Using $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, and the Quotient Rule, we get

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \end{aligned}$$

[5] 8. Find the constant k so that the tangent line to the curve $y = \dfrac{4k}{x} + 8$ at the point $x = 2$ has slope 5.

Solution. We want y' to equal 5 at $x = 2$. We get

$$y = (4k)x^{-1} + 8,$$

so

$$y' = (4k)(-1)x^{-2},$$

so when $x = 2$, y' equals $-4k/2^2 = -k$. Thus we want $-k = 5$, so $k = \mathbf{-5}$.

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