90 Minute Exam (Multiple Choice) ~~249~~ Solns. AMAT 217

1. Find $\lim_{x\to 2}\left(\dfrac{x^3-8}{x-2}\right)$.

(A) 12
(B) 4
(C) ∞
(D) does not exist
(E) 8

Factor: $x^3-8 = x^3-2^3 = (x-2)(x^2+2x+4)$

2. Find $\lim_{t\to 0}\left(\dfrac{\sqrt{t^2+4}-2}{t}\right)$.

(A) -1
(B) does not exist
(C) 1
(D) 2
(E) 0

Rationalize:

$$\left(\frac{\sqrt{t^2+4}-2}{t}\right)\frac{(\sqrt{t^2+4}+2)}{(\sqrt{t^2+4}+2)}$$

$$= \frac{t^2+4-4}{t\,\sqrt{t^2+4}+2} = \frac{t}{\sqrt{t^2+4}+2}$$

3. Find $\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-3x}\right)$.

(A) ∞
(B) -3
(C) $-2\sqrt{3}$
(D) 3
(E) $2\sqrt{3}$

Rationalize: $\left(\underbrace{\sqrt{x^2+3x}}_{A}-\underbrace{\sqrt{x^2-3x}}_{B}\right)\dfrac{\left(\underbrace{\sqrt{x^2+3x}}_{A}+\underbrace{\sqrt{x^2-3x}}_{B}\right)}{\left(\sqrt{x^2+3x}\right)+\left(\sqrt{x^2-3x}\right)}$

Since $(A-B)(A+B) = A^2-B^2$ we get $\dfrac{6x}{\sqrt{x^2+3x}+\sqrt{x^2-3x}}$. As x gets large we get in the limit

$$\lim_{x\to\infty}\frac{6x}{2\sqrt{x^2}} = \lim_{x\to\infty}\frac{6x}{2x} = 3$$

Note: for $x\to-\infty$ we would get $\dfrac{6x}{-2x} = -3$

Since: $\sqrt{x^2} = x$ if $x \geq 0$
$\sqrt{x^2} = -x$ if $x < 0$

4. Find $\lim_{x\to\infty}\dfrac{5x^2-2x-3}{(2x+3)(3x+2)}$.

(A) $-1/2$
(B) ∞
(C) $-\infty$
(D) $5/6$
(E) $5/13$

In the limit we get $\dfrac{5x^2 \ldots}{6x^2 \ldots}$

5. Find $\lim_{x\to-\infty}\left(\dfrac{3x+7}{\sqrt{4x^2+5}}\right)$.

(A) $7/\sqrt{5}$
(B) $-3/\sqrt{5}$
(C) 0
(D) $3/2$
(E) $-3/2$

As x gets large (and negative) we are looking at $\dfrac{3x}{\sqrt{4x^2}}$

$$= \frac{3x}{\sqrt{4}\,\sqrt{x^2}} = \frac{3x}{2(-x)} = -\frac{3}{2}$$

(see soln. for #3)

6. The derivative of $4x^3 + 3x^2 + 2x + 1$ is

(A) $x^4 + x^3 + x^2 + x$
(B) $4x^2 + 3x + 2$
(C) $(x^5 - 1)/(x - 1)$
(D) $x^4 + x^3 + x^2 + x + C$
(E) $12x^2 + 6x + 2$

$$\frac{d}{dx}\left(4x^3+3x^2+2x+1\right)$$
$$= \frac{d}{dx}(4x^3) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(1)$$
$$= 4\frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x) + 0$$
$$= 12x^2 + 6x + 2$$

7. The derivative of $\dfrac{x^2}{x^4+1}$ is

(A) $\dfrac{1}{2x^2}$
(B) $\dfrac{6x^5 + 2x}{(x^4+1)^2}$
(C) $\dfrac{2x^5 - 2x}{(x^4+1)^2}$
(D) $\dfrac{2x - 4x^3}{(x^4+1)}$
(E) $\dfrac{2x - 2x^5}{(x^4+1)^2}$

$$\frac{d}{dx}\left(\frac{x^2}{x^4+1}\right) = \frac{\left(\frac{d}{dx}(x^2)\right)(x^4+1) + x^2\frac{d}{dx}(x^4+1)}{(x^4+1)^2}$$
$$= \frac{(2x)(x^4+1) + x^2(4x^3)}{(x^4+1)^2}$$
$$= \frac{6x^5 + 2x}{(x^4+1)^2}$$

8. The derivative of $x^2\tan(3x)$ is

(A) $2x\tan(3x) + 3x^2\tan(3x)\sec(3x)$
(B) $6x\sec^2(3x)$
(C) $x^2\cot(3x)$
(D) $2x\tan(3x) + 3x^2\sec^2(3x)$
(E) $6x\tan(3x)$

$$\frac{d}{dx}\left[x^2\tan(3x)\right]$$
$$= \left[\frac{d}{dx}(x^2)\right]\tan(3x) + x^2\frac{d}{dx}\left(\tan(3x)\right)$$
$$= 2x\tan(3x) + x^2\sec^2(3x)\,3$$
$$= 2x\tan(3x) + 3x^2\sec^2(3x)$$

9. The derivative of $\ln(x^3)$ is

(A) $3/x$
(B) $x/\ln(x)$
(C) $3x^2\ln(x^3)$
(D) $3x^2/\ln(x^3)$
(E) $3x^2/\ln(x)$

10. The derivative of $y = x^{\cos(x)}$ is

(A) $(\cos(x)/x - \sin(x)\ln(x))x^{\cos(x)}$
(B) $e^x x^{\cos(x)}$
(C) $\ln(x)x^{\cos(x)}$
(D) $-\sin(x)x^{\cos(x)-1}$
(E) $(\sin(x)/x - \ln(x))x^{\cos(x)}$

16. The statement $\lim_{x \to a} f(x) = L$ means

(A) $f(a) = L$
(B) if $x < a$ then $f(x) < L$, and if $x > a$ then $f(x) > L$
(C) the function f is continuous at $x = a$
(D) For x close to a, $f(x)$ will be close to L
(E) for x close to a, but $x \neq a$, $f(x)$ will be close to L

When calculating the limit, x is never allowed to be actually equal to a. If f is continuous at $x = a$ then $f(a) = \lim_{x \to a} (f(x))$ and L is finite

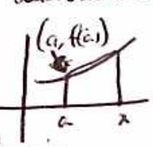
17. Which function is not continuous on its domain?

(A) $x/|x|$
(B) $\ln x$
(C) $|x|$
(D) $1/(x^2 - 1)$
(E) $\lfloor x \rfloor$

18. If $f(x) = x^3$, which of the following limits equals $f'(2)$?

(A) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$
(B) $\lim_{h \to 0} \frac{(2-h)^3 + h^3}{h}$
(C) $\lim_{h \to 0} \frac{3h + 3h^2 + h3}{h}$
(D) $\lim_{x \to 2} \frac{(2+h)^3 - h^3}{h}$
(E) $\lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h}$

One version of a formula for the derivative at $x = a$ is $\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$



In class we use, usually, the following formula: Let $x = a + h$. Then, as $x \to a$ $h \to 0$. So we get for $f'(a)$ the formula

$$f'(a) = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{(2+h)^3 - 2^3}{h} \right], \text{ since } a = 2$$

19. The function $f(x) = \sqrt{x^2 - 1}$ has natural domain:

(A) $[-1, 1]$
(B) $[1, \infty)$
(C) $(-\infty, -1] \cup [1, \infty)$
(D) $(-\infty, \infty)$
(E) $[0, 1]$

The natural domain is the set of all x for which $x^2 - 1 \geq 0$ ie $x^2 \geq 1$. So, either $x \geq 1$ OR $x \leq -1$. This gives $[1, \infty)$ together with $[-\infty, -1]$ ie $[1, \infty) \cup (-\infty, -1]$

20. The line $y = 3x + 1$ is tangent to some curve at point $(\frac{1}{3}, 2)$. The normal line, at the same point, is

(A) $y = -\frac{x}{3} + \frac{19}{9}$
(B) $y = 3x - 1$
(C) $y - 2 = \frac{1}{3}(x - \frac{1}{3})$
(D) $y = -\frac{x}{3} + 1$
(E) None of these.

Normal = perpendicular = at right angles.
The equation of the tangent line is $y - 2 = (\text{slope})(x - \frac{1}{3})$
slope is $\frac{dy}{dx}\Big|_{x=\frac{1}{3}} = m$, say. So we have $y - 2 = m(x - \frac{1}{3})$ ie $y = mx - \frac{m}{3} + 2$
This is the same line as the line $y = 3x + 1$, so $m = 3$. The normal line has slope $-\frac{1}{3}$.
Eqn is: $y - 2 = -\frac{1}{3}(x - 1/3)$

21. The value of $\tan(5\pi/4)$ is

(A) $\sqrt{2}/2$
(B) $-\sqrt{2}/2$
(C) 1
(D) $\sqrt{3}$
(E) -1

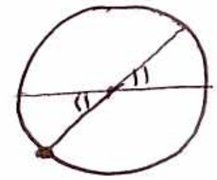
$\frac{5\pi}{4} = \pi + \frac{\pi}{4}$

$\sin\left(\frac{5\pi}{4}\right) = -\sin\frac{\pi}{4}$

$\cos\left(\frac{5\pi}{4}\right) = -\cos\frac{\pi}{4}$

$\tan\left(\frac{5\pi}{4}\right) = \frac{\sin\left(\frac{5\pi}{4}\right)}{\cos(5\pi/4)} = \frac{-\sin\frac{\pi}{4}}{-\cos\pi/4}$

$= \frac{\sin\frac{\pi}{4}}{\cos\pi/4} = \tan(\pi/4) = 1$



22. How many solutions are there to the equation $4(2^x)^x = 8^x$.

(A) 1
(B) 3
(C) 0
(D) 2
(E) more than 3

23. Use the laws of logs to simplify $\log_{15} 75 + \log_{15} 3$.

(A) 2
(B) $\log_{15} 78$
(C) $5 + 1/5$
(D) $\log(5) + \log(1/5)$
(E) $\log_{30} 225$

24. Evaluate $\lim_{x\to-\infty} \tanh(x)$.

(A) $-\infty$
(B) $+1$
(C) $+\infty$
(D) -1
(E) 0

25. Evaluate $\sinh(\ln 3)$.

(A) $(e^3 - e^{-3})/2$
(B) $5/3$
(C) $4/3$
(D) $\cosh(-\ln 3)$
(E) $(\ln(3) + \ln(1/3))/2$

31. The tangent line to curve $y = \cos(2x)$ at point $(\frac{\pi}{6}, \frac{1}{2})$ is

(A) $y = -\sqrt{3}x + \frac{1}{2} + \frac{\sqrt{3}\pi}{6}$
(B) $y = \sqrt{2}x + \frac{1}{2} - \frac{\sqrt{2}\pi}{6}$
(C) $y = \sqrt{3}x + \frac{1}{2} - \frac{\sqrt{3}\pi}{6}$
(D) $y = \frac{1}{2} - 2\sin(2x)\cdot(x - \frac{\pi}{6})$
(E) $y = -\sqrt{2}x + \frac{1}{2} + \frac{\sqrt{2}\pi}{6}$

32. Given the curve $x^2y^3 = 4x - y$ which passes through the point $P = (3, 1)$, the slope at P equals

(A) $(4 - 2xy^3)/(3x^2y^2 + 1)$
(B) $3/(3x^2y^2 + 1)$
(C) $-25/14$
(D) $4/(3x^2y^2 + 1)$
(E) $-1/14$

33. The curve in the previous question also passes through $(0, 0)$. The Mean Value Theorem says

(A) for some x between 0 and 3, $\frac{dy}{dx} = -3$
(B) for some x between 0 and 1, $\frac{dy}{dx} = \frac{1}{3}$
(C) for some x near 0, $\frac{dy}{dx} = 3$
(D) for some y between 0 and 1, $\frac{dy}{dx} = 3$
(E) for some x between 0 and 3, $\frac{dy}{dx} = \frac{1}{3}$

34. Given the initial value problem $y' = x^{1/3}, y(1) = \frac{11}{4}$, then the solution $y = \frac{3}{4}x^{4/3} + C$ has constant

(A) $C = 1$
(B) $C = -2$
(C) $C = 2$
(D) $C = 0$
(E) $C = 11/4$

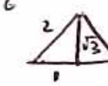
35. The inverse to function $f(x) = \dfrac{x}{x+1}$ is

(A) $f^{-1}(x) = \frac{x+1}{x}$
(B) $f^{-1}(x) = \frac{1}{(x+1)^2}$
(C) $f^{-1}(y) = \frac{x}{x+1}$
(D) $f^{-1}(x) = \frac{x}{1-x}$
(E) $f^{-1}(y) = \frac{y}{y+1}$

$(\frac{\pi}{6}, \frac{1}{2})$ is a point on the tangent line.

The slope is $\frac{dy}{dx}\Big|_{x=\pi/6} = \frac{d(\cos(2x))}{dx}\Big|_{x=\frac{\pi}{6}}$

$= -2\sin 2x\Big|_{x=\frac{\pi}{6}} = -2\sin\frac{2\pi}{6}$



$= -2\sin(\frac{\pi}{3}) = -2(\frac{\sqrt{3}}{2}) = -\sqrt{3}$

The equation of the tangent line is

$y - \frac{1}{2} = -\sqrt{3}(x - \pi/6)$

$\frac{d}{dx}(x^2y^3) = \frac{d}{dx}(4x - y)$

$[\frac{d}{dx}(x^2)]\,y^3 + x^2\frac{d}{dx}(y^3) = \frac{d(4x)}{dx} - \frac{d(y)}{dx}$

$\therefore 2xy^3 + (x^2)\,3y^2\frac{dy}{dx} = 4 - \frac{dy}{dx}$

$\therefore \frac{dy}{dx}[3x^2y^2 + 1] = 4 - 2xy^3$

$\frac{dy}{dx}\Big|_{(3,1)} = \frac{4 - 2xy^3}{3x^2y^2 + 1}\Big|_{(3,1)}$

$= \frac{4 - 2(3)}{3\ 3^2 + 1} = \frac{-2}{28}$

$= -\frac{1}{14}$