# MAT 249 - Quizz 4

Winter 2007 - Lecture 08 - L. Nguyen Van Thé

Labs 31, 32 - 14:00, Wednesday, March 7, 2007

Duration: 30 minutes - Points: 8

## NO CALCULATORS

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**Name:** **ID:**

**Grade:** **/8**

**Definitions and theorems:** [2 pts]
**1-** [1 pt] Let $a, b \in \mathbb{R}$, $a < b$, $x_0 \in ]a, b[$, $f$ a function defined on $]a, b[$. What does it mean for $f$ to be continuous at $x_0$ (In other words, give the definition of continuity of $f$ at $x_0$)?

$f$ is continuous at $x_0$ iff $f$ has a finite limit $L$ at $x_0$ and $L = f(x_0)$.

**2-** [1 pt] State the mean value theorem.

Let $a, b \in \mathbb{R}$, $a < b$, $f$ a function defined on $[a, b]$, continuous on $[a, b]$, and differentiable on $]a, b[$. Then there is $c \in ]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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**Problem 2:** [3 pts]

Consider the functions $f$ and $g$ defined on $]0,\pi[$ by:

$$f(x) = \frac{\sin x}{x} \qquad g(x) = \sqrt{f(x)}.$$

**1-** [1pt] Explain why $f$ and $g$ do not have any problem of definition on $]0,\pi[$.

Remark: This question can be reformulated as follows: Consider the functions $f$ and $g$ defined on $]0,\pi[$ as above. Show that $f$ and $g$ have domain $]0,\pi[$.

$f$ is the quotient of two functions that are defined on $]0,\pi[$, and the denominator function does not take the value 0 when $x$ is in $]0,\pi[$. Thus, $f$ has no definition problem on $]0,\pi[$.

On the other hand, $g$ is obtained by taking the square root of $f$ on $]0,\pi[$. We just proved that $f$ is defined on $]0,\pi[$, so the expression under the square root is always defined on $]0,\pi[$. We consequently simply need to make sure that this expression is $\geqslant 0$. But this is true since for $x \in ]0,\pi[$, $\sin x > 0$ and $x > 0$.

**2-** [1pt] Explain why $f$ is differentiable on $]0,\pi[$ and compute its derivative.

$f$ is differentiable on $]0,\pi[$ because it is the quotient of two differentiable functions on $]0,\pi[$ (namely $h$ and $i$ defined on $]0,\pi[$ by $h(x) = \sin x$ and $i(x) = x$) and the function in the denominator does not take the value 0 on $]0,\pi[$.

The derivative of $f$ is obtained by applying the quotient rule: Let $x \in ]0,\pi[$. Then

$$f'(x) = \frac{\cos x \cdot x - \sin x}{x^2}.$$

**3-** [1pt] Explain why $g$ is differentiable on $]0,\pi[$ and compute its derivative.

$g$ is differentiable on $]0,\pi[$ because it is obtained by composing the square root function, which is differentiable on $]0,+\infty[$, and the function $f$, which is differentiable on $]0,\pi[$ and takes values in $]0,+\infty[$.

Remark: A problem would have shown up if $f$ had taken the value 0 on $]0,\pi[$ because the square root function is not differentiable at 0. However, $f$ does not take the value 0 on $]0,\pi[$ so this does not cause any harm here.

The derivative of $g$ can be obtained by applying the chain rule. Let $x \in ]0,\pi[$. Then

$$g'(x) = \frac{1}{2\sqrt{f(x)}} f'(x) = \frac{\cos x \cdot x - \sin x}{2x^2\sqrt{\frac{\sin x}{x}}}.$$

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**Problem 3:** [3 pts]

Let $h$ be the function defined by

$$h(x) = -\frac{1}{4}x^4 - \frac{2}{3}x^3 + 3x - 6.$$

**1-** [1pt] Explain why $h$ is defined and differentiable on $\mathbb{R}$ and compute its derivative.

$h$ is defined and differentiable on $\mathbb{R}$ because it is a polynomial function. Its derivative is defined on $\mathbb{R}$ by $h'(x) = -x^3 - 2x^2 + 3$.

**2-** [1pt] Study the sign of $h'$. Express your result in a sign table.

To study the sign of $h'$, we need to factor it. As its degree is higher than 2, the only thing we can start with is to find a root directly. Testing the usual simple values $-2, -1, 0, 1, 2$, we find that 1 is a root. Therefore, $h'(x)$ can be factored by $(x - 1)$. After having applied one of the standard factorization methods, we find that $h'(x) = -(x-1)(x^2+3x+3)$. Call $P(x)$ the right factor. $P$ is a quadratic polynomial function. To know whether it can be factored, we compute its discriminant $\Delta$. Here $\Delta = 3^2 - 4 \cdot 1 \cdot 3 = 9 - 12 < 0$. Thus, $P$ has no real root. Therefore, it cannot be factored further and has a constant sign on $\mathbb{R}$. Since $P(0) = 3 > 0$, $P(x)$ is always positive. The sign of $h'(x)$ consequently only depends on the sign of $(x + 1)$, and the corresponding sign table is:

| $x$ | $-\infty$ | | 1 | | $+\infty$ |
|---|---|---|---|---|---|
| $x - 1$ | | - | 0 | + | |
| $h'(x)$ | | + | 0 | - | |

**3-** [1pt] Using the result of question 2, find the intervals of increase and decrease of $h$.

According to the result of the previous question, $h$ increases on $]-\infty, 1]$ and decreases on $[1, +\infty[$.