MATHEMATICS 249 L04 MIDTERM November 10, 2009

[5] 1. Find $\lim_{x\to -2}\left(\dfrac{x+2}{\sqrt{x+11}-3}\right)$. Do not use l'Hôpital's Rule.

Solution.

$$\begin{aligned}\lim_{x\to -2}\left(\frac{x+2}{\sqrt{x+11}-3}\right) &= \lim_{x\to -2}\left(\frac{(x+2)(\sqrt{x+11}+3)}{(\sqrt{x+11}-3)(\sqrt{x+11}+3)}\right)\\ &= \lim_{x\to -2}\left(\frac{(x+2)(\sqrt{x+11}+3)}{(x+11)-9}\right)\\ &= \lim_{x\to -2}\left(\frac{(x+2)(\sqrt{x+11}+3)}{x+2}\right)\\ &= \lim_{x\to -2}(\sqrt{x+11}+3) = \sqrt{9}+3 = 3+3 = \mathbf{6}.\end{aligned}$$

[5] 2. Find $\lim_{x\to 0}\left(\dfrac{e^x+2x-1}{e^{2x}+3x-1}\right)$.

Solution. $\lim_{x\to 0}\left(\dfrac{e^x+2x-1}{e^{2x}+3x-1}\right)$ is of the form $\dfrac{e^0+0-1}{e^0+0-1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$, so we can use l'Hôpital's Rule. We get

$$\begin{aligned}\lim_{x\to 0}\left(\frac{e^x+2x-1}{e^{2x}+3x-1}\right) &= \lim_{x\to 0}\left(\frac{(e^x+2x-1)'}{(e^{2x}+3x-1)'}\right)\\ &= \lim_{x\to 0}\left(\frac{e^x+2}{e^{2x}\cdot 2+3}\right)\\ &= \frac{e^0+2}{e^0\cdot 2+3} = \frac{1+2}{2+3} = \frac{\mathbf{3}}{\mathbf{5}}\ .\end{aligned}$$

[5] 3. Find y' where $y = \cos^2(\sqrt{x})$.

Solution. By the Chain Rule,

$$y' = \frac{d}{dx}\cos^2(\sqrt{x}) = 2\cos\sqrt{x}\cdot(-\sin\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\ .$$

[6] 4. Use implicit differentiation to find $\dfrac{dy}{dx}$ where $x^2y = \tan(y^2-4y)$.

Solution. We get

$$\frac{d}{dx}(x^2y) = \frac{d}{dx}\tan(y^2-4y)\ ,$$

$$2xy + x^2\cdot\frac{dy}{dx} = \sec^2(y^2-4y)\left(2y\cdot\frac{dy}{dx} - 4\cdot\frac{dy}{dx}\right),$$

and so

$$2xy = \sec^2(y^2 - 4y)(2y-4)\frac{dy}{dx} - x^2 \cdot \frac{dy}{dx} \ ,$$

$$2xy = \left(\sec^2(y^2 - 4y)(2y-4) - x^2\right)\frac{dy}{dx} \ ,$$

and finally

$$\frac{dy}{dx} = \frac{2xy}{\sec^2(y^2-4y)(2y-4) - x^2} \ .$$

[5] 5. Find $\dfrac{d}{dx}\left(\dfrac{\ln(3x)}{3-4x}\right)$.

Solution. By the Quotient Rule,

$$\frac{d}{dx}\left(\frac{\ln(3x)}{3-4x}\right) = \frac{(3-4x)\cdot\frac{1}{3x}\cdot 3 - \ln(3x)(-4)}{(3-4x)^2} \ .$$

[5] 6. USE THE LIMIT DEFINITION OF DERIVATIVE to find $\dfrac{d}{dx}(4-x^2)$.

Solution. We get

$$\begin{aligned}
\frac{d}{dx}(4-x^2) &= \lim_{h\to 0}\left(\frac{4-(x+h)^2-(4-x^2)}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{4-(x^2+2xh+h^2)-(4-x^2)}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{4-x^2-2xh-h^2-4+x^2}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{-2xh-h^2}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{h(-2x-h)}{h}\right) \\
&= \lim_{h\to 0}(-2x-h) = -2x.
\end{aligned}$$

7. Let $\quad f(x) = \begin{cases} k-5x & \text{if } x \le 2, \\ 7-x^2 & \text{if } x > 2, \end{cases}$ $\quad$ where k is a constant.

[4] (a) Find and simplify the equation of the tangent line to the graph $y = f(x)$ at the point on the graph where $x = 3$.

Solution. Since $3 > 2$, we must use $f(x) = 7 - x^2$ to find the tangent line at $x = 3$. Thus $f'(x) = -2x$ at $x = 3$, so the slope of the tangent line at $x = 3$ will be $-2(3) = -6$.

Also, when $x = 3$, $y = f(3) = 7 - 3^2 = -2$. Thus the equation of the tangent line will be $y - (-2) = -6(x - 3)$, which simplifies to

$$y = -6x + 16.$$

[2 + 3] (b) Find the constant k so that f is continuous at $x = 2$. When k equals this value, is f also differentiable at $x = 2$? Explain.

Solution. For f to be continuous at $x = 2$, we need that the one-sided limits $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ are equal (so that the two parts of the curve will "hook together" at $x = 2$).

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (k - 5x) = k - 10$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (7 - x^2) = 7 - 4 = 3,$$

so we need $k - 10 = 3$, which means $k = \mathbf{13}$.

When $k = 13$, our function is

$$f(x) = \begin{cases} 13 - 5x & \text{if } x \le 2, \\ 7 - x^2 & \text{if } x > 2, \end{cases}$$

so

$$f'(x) = \begin{cases} -5 & \text{if } x < 2, \\ -2x & \text{if } x > 2. \end{cases}$$

Thus for f to be differentiable at $x = 2$, we would need that $\lim_{x\to 2^+} f'(x) = -5$, so that the two parts of the curve will "hook together smoothly" at $x = 2$, that is, the tangent line at $x = 2$ from either side would have slope -5. However,

$$\lim_{x\to 2+} f'(x) = \lim_{x\to 2+} (-2x) = -4 \ne -5,$$

so f is **not** differentiable at $x = 2$.

[40]