MATHEMATICS 249 L03 MIDTERM SOLUTIONS March 23, 2010

[5] 1. Find $\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{3}{x^2-3x}\right)$. Do not use l'Hôpital's Rule.

Solution. We get

$$\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{3}{x^2-3x}\right)=\lim_{x\to 3}\left(\frac{x}{x(x-3)}-\frac{3}{x(x-3)}\right)=\lim_{x\to 3}\left(\frac{x-3}{x(x-3)}\right)=\lim_{x\to 3}\left(\frac{1}{x}\right)=\frac{1}{3}\,.$$

[5] 2. Find $\lim_{x\to 1}\left(\frac{e^x-e}{\ln(2x-1)}\right)$.

Solution. Since

$$\lim_{x\to 1}(e^x-e)=e^1-e=e-e=0$$

and

$$\lim_{x\to 1}(\ln(2x-1))=\ln(2\cdot 1-1)=\ln(2-1)=\ln 1=0,$$

the given limit is of the form 0/0, so we can use l'Hôpital's Rule. Using it we get

$$\begin{aligned}\lim_{x\to 1}\left(\frac{e^x-e}{\ln(2x-1)}\right) &= \lim_{x\to 1}\left(\frac{(e^x-e)'}{(\ln(2x-1))'}\right)=\lim_{x\to 1}\left(\frac{e^x-0}{\frac{1}{2x-1}\cdot 2}\right)=\lim_{x\to 1}\left(\frac{e^x(2x-1)}{2}\right)\\ &= \frac{e^1(2\cdot 1-1)}{2}=\frac{e}{2}\,.\end{aligned}$$

[5] 3. Find y' where $y=\sqrt{x\sin 4x}$.

Solution. Using the Chain Rule and the Product Rule, we get

$$y'=(\sqrt{x\sin 4x})'=\frac{1}{2\sqrt{x\sin 4x}}\big(1\cdot\sin 4x+x\cos 4x\cdot 4\big).$$

[5] 4. Use implicit differentiation to find $\frac{dy}{dx}$ where $xy^2=(\ln x)^2+4y$.

Solution. We get

$$\frac{d}{dx}(xy^2)=\frac{d}{dx}\big((\ln x)^2+4y\big),$$

$$1\cdot y^2+x\cdot 2y\frac{dy}{dx}=2\ln x\cdot\frac{1}{x}+4\frac{dy}{dx},$$

$$2xy\frac{dy}{dx}-4\frac{dy}{dx}=\frac{2}{x}\ln x-y^2,$$

$$(2xy-4)\frac{dy}{dx}=\frac{2}{x}\ln x-y^2,$$

and finally

$$\frac{dy}{dx} = \frac{\frac{2}{x}\ln x - y^2}{2xy - 4}\ .$$

[5] 5. Find $\dfrac{d}{dx}\left(\dfrac{5-3x}{\tan(x^2-5)}\right)$.

Solution. Using the Quotient Rule and the Chain Rule, we get

$$\frac{d}{dx}\left(\frac{5-3x}{\tan(x^2-5)}\right) = \frac{\tan(x^2-5)(-3) - (5-3x)\sec^2(x^2-5)\cdot 2x}{\tan^2(x^2-5)}\ .$$

[5] 6. USE THE LIMIT DEFINITION OF DERIVATIVE to find $\dfrac{d}{dx}(\sqrt{7x})$.

Solution. We get

$$\begin{aligned}
\frac{d}{dx}(\sqrt{7x}) &= \lim_{h\to 0}\left(\frac{\sqrt{7(x+h)}-\sqrt{7x}}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{(\sqrt{7x+7h}-\sqrt{7x})(\sqrt{7x+7h}+\sqrt{7x})}{h(\sqrt{7x+7h}+\sqrt{7x})}\right) \\
&= \lim_{h\to 0}\left(\frac{7x+7h-7x}{h(\sqrt{7x+7h}+\sqrt{7x})}\right) \\
&= \lim_{h\to 0}\left(\frac{7h}{h(\sqrt{7x+7h}+\sqrt{7x})}\right) \\
&= \lim_{h\to 0}\left(\frac{7}{\sqrt{7x+7h}+\sqrt{7x}}\right) \\
&= \frac{7}{\sqrt{7x}+\sqrt{7x}} = \frac{7}{2\sqrt{7x}}\ .
\end{aligned}$$

[6] 7. (a) Find all critical points of the function $f(x) = x^4 - 8x^2$.

Solution. We first find that $f'(x) = 4x^3 - 16x$ (which is always defined), so $f'(x) = 0$ when $4x^3 - 16x = 0$, which can be written $4x(x^2-4) = 0$ and then $4x(x-2)(x+2) = 0$. Thus the critical points are $x = 0, x = 2$ and $x = -2$.

(b) Find the equation of the tangent line to the curve $y = x^4 - 8x^2$ at the point where $x = -1$.

Solution. When $x = -1$,

$$y = (-1)^4 - 8(-1)^2 = 1 - 8 = -7$$

and from part (a)

$$y' = 4(-1)^3 - 16(-1) = -4 + 16 = 12,$$

so the slope of the tangent line at the point where $x = -1$ is $m = 12$. Therefore the equation of the tangent line is

$$y - (-7) = 12(x - (-1)),$$

which simplifies to $y = 12x + 5$.

[4] 8. Find the constant k so that the function $f(x) = \begin{cases} kx + 2 & \text{if } x < 1 \\ x^3 + 4x + 5 & \text{if } x \geq 1 \end{cases}$ is continuous at $x = 1$. When k equals this value, is f also differentiable at $x = 1$? Explain.

Solution. For f to be continuous at $x = 1$, we need

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1).$$

Well,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (kx + 2) = k + 2$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^3 + 4x + 5) = 1^3 + 4 \cdot 1 + 5 = 10 = f(1),$$

so we need $k + 2 = 10$, thus $k = 8$.

If we put $k = 8$, then for f to be differentiable at $x = 1$ we will need

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x).$$

Well,

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} (8x + 2)' = \lim_{x\to 1^-} 8 = 8$$

and

$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} (x^3 + 4x + 5)' = \lim_{x\to 1^+} (3x^2 + 4) = 3 \cdot 1^2 + 4 = 7 \neq 8,$$

so

$$\lim_{x\to 1^-} f'(x) \neq \lim_{x\to 1^+} f'(x)$$

and therefore f is **not** differentiable at $x = 1$. This means: although (when $k = 8$) the two pieces of the graph of $y = f(x)$ do hook together at $x = 1$, they don't hook together smoothly.

$\overline{[40]}$