The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #2

Solution.

RECALL:the types of limits :"

$\frac{1}{\pm\infty}" = 0$ $\quad$ $"\frac{1}{0^+}"(\frac{1}{\text{small positive numbers}}) = +\infty$ $\quad$ $"\frac{1}{0^-}"(\frac{1}{\text{small negativee numbers}}) = -\infty$

BUT $\quad "\frac{0}{0}", " \pm\frac{\infty}{\infty}", "\infty - \infty", "0 \cdot \pm\infty"$ are indetermined types it means the result could be any number or infinity thus we have to simplify

For 1a) $\quad$ For $f(x) = \dfrac{1}{1-x}\left(1 - \dfrac{4}{x+3}\right)$

the type of the limit is $"\frac{0}{0}"$as $x \to 1$ so we have to simplify

$$f(x) = \frac{1}{1-x} \cdot \frac{x+3-4}{x+3} = \frac{-(1-x)}{(1-x)(x+3)} = \frac{-1}{x+3} \text{ for any } x \neq 1, -3.$$

As $x \to 1$ the limit is $L = \frac{-1}{4}$.

For 1b)

as $x \to -3^+ \qquad x + 3 > 0$

we can use the simplification from above and the type of the limit is $"\frac{-1}{0^+}"$ so the limit is $-\infty$. $\quad$ OR

from the original formula the limit is $L = \frac{1}{4} \cdot \left(1 - "\frac{4}{0^+}"\right) = -\infty$

For 1c)

as $x \to +\infty$.

From the simplified formula the type is $"\frac{-1}{\infty}"$ so the limit is 0.

OR

from the original the type is $"\frac{1}{-\infty}" \cdot \left(1 - "\frac{4}{\infty}"\right) = 0 \cdot (1-0) = 0$.

For 2)

For $f(x) = \sqrt{9-x^2}$ and $g(x) = \dfrac{3}{x-1}$ find the domains

for $D_f \qquad$ solve $9 - x^2 \geq 0 \qquad \to \qquad (3-x)(3+x) \geq 0$

parabola open down, above the x-axis between roots

OR by split point method

split points are $x = \pm 3$, testing $- -^{neg} - -_{-3} - - -^{pos} - - -_{3} - -^{neg} - - -$

so $D_f = [-3, 3] \qquad\qquad D_g = \{x \neq 1\}$ since $x - 1 \neq 0$

Now,

$$g \circ g(x) = g(g(x)) = \frac{3}{(...) - 1} = \frac{3}{\frac{3}{x-1} - 1} = \frac{3}{\frac{3-(x-1)}{x-1}} = \frac{3(x-1)}{4-x}$$

we must start in D_g i.e. $x \neq 1$ and we have to guarantee that $4 - x \neq 0$ so $x \neq 4$ together $\qquad D_{g \circ g} = \{x \neq 1, 4\}$

then

$$f \circ g(x) = \sqrt{9 - (..)^2} = \sqrt{9 - \frac{9}{(x-1)^2}} = \sqrt{9 \cdot \left(1 - \frac{1}{(x-1)^2}\right)} = 3 \cdot \sqrt{\frac{x^2-2x+1-1}{(x-1)^2}} = 3\sqrt{\frac{x(x-2)}{(x-1)^2}}$$

we must start in $\mathrm{D}_g, x \neq 1$ and quarantee that $\quad \dfrac{x(x-2)}{(x-1)^2} \geq 0$

split points are $x = 0, 2, 1$

testing $--^{pos}-_0--^{neg}--_1--^{neg}--_2-^{pos}---$so $\quad D_{f\circ g}=(-\infty,0]\cup[2,\infty)$.

For 3)

For $g(x)=\dfrac{4}{2x-8}$ and $f(x)=\sqrt{x^2-9}$

$D_g=\{x\neq 4\}$ and $D_f=(-\infty,-3]\cup[3,+\infty)$ since we have to solve: $x^2-9\geq 0$

$(x-3)(x+3)\geq 0$ parabola open up with roots $x=\pm 3$

$OR \quad x^2\geq 9 \quad \sqrt{x^2}=|x|\geq 3$

Now, $g\circ g(x)=\dfrac{4}{2\,()-8}=\dfrac{2\cdot 2}{2\left[\left(\frac{4}{2x-8}\right)-4\right]}=\dfrac{2}{\frac{4-8x+32}{2x-8}}=2\cdot\dfrac{2x-8}{36-8x}=\dfrac{4\,(x-4)}{4\,(9-2x)}=\dfrac{x-4}{9-2x}$

for $x\neq 4$ and $x\neq\frac{9}{2}$ thus $D_{g\circ g}=(-\infty,4)\cup\left(4,\frac{9}{2}\right)\cup\left(\frac{9}{2},+\infty\right)$.

then

$g\circ f(x)=\dfrac{4}{2\,()-8}=\dfrac{4}{2\sqrt{x^2-9}-8}=$ you can simplify

$=\dfrac{2}{\sqrt{x^2-9}-4}\cdot\dfrac{\sqrt{x^2-9}+4}{\sqrt{x^2-9}+4}=\dfrac{2\left(\sqrt{x^2-9}+4\right)}{x^2-9-4^2}=\dfrac{2\left(\sqrt{x^2-9}+4\right)}{x^2-25}$

we must start with $x\in D_f$ and that new denominator must be non-zero

$\sqrt{x^2-9}-4\neq 0, \sqrt{x^2-9}\neq 4$, so $x^2-9\neq 4^2, x^2\neq 25$

OR

using the simplification $x^2-25\neq 0$ i.e. $x\neq\pm 5$,together with D_f

$D_{g\circ f}=(-\infty,-5)\cup(-5,-3]\cup[3,5)\cup(5,+\infty)$. .

For 4a)

for $f(x)=\dfrac{1-4x^2}{6x^2-5x+1}$ as $x\to-\infty$ the type is "$\frac{-\infty}{\infty}$" so

divide top and bottom by x^2 :

$\lim\limits_{x\to-\infty}\dfrac{\frac{1}{x^2}-4}{6-\frac{5}{x}+\frac{1}{x^2}}=\dfrac{0-4}{6-0+0}=-\frac{4}{6}=-\dfrac{2}{3}$ (since "$\frac{1}{\pm\infty}$" $=0$)

For 4b)

as $x\to\frac{1}{2}$ the type is "$\frac{0}{0}$" so factorize the polynomials

$\lim\limits_{x\to\frac{1}{2}}\dfrac{(1-2x)(1+2x)}{(2x-1)(3x-1)}=\lim\limits_{x\to\frac{1}{2}}\dfrac{-(1+2x)}{3x-1}=\frac{-2}{\frac{1}{2}}=-4.$

For 4c) as $x\to\frac{1}{3}^-$

we can use the simplification from above then the type is "$\frac{neg\#}{0^-}$" since $x<\frac{1}{3}$ so $3x-1<0$

$\lim\limits_{x\to\frac{1}{3}^-}\dfrac{-(1+2x)}{3x-1}=$ "$\frac{-\frac{5}{3}}{0^-}$" $=$ "$\frac{1}{0^+}$" $=+\infty$

For 5a)

For $\dfrac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}$ as $x\to 3^+$

$\lim\limits_{x\to 3^+}\dfrac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}=$ "$\frac{0}{0}$" $=\lim\limits_{x\to 3^+}\dfrac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}\cdot\dfrac{\sqrt{3x}+3}{\sqrt{3x}+3}=\lim\limits_{x\to 3^+}\dfrac{3x-3^2}{\sqrt{2x^2-6x}}\cdot\dfrac{1}{\sqrt{3x}+3}=$

$=\lim\limits_{x\to 3^+}\dfrac{3(x-3)}{\sqrt{2x(x-3)}}\cdot\dfrac{1}{\sqrt{3x}+3}=\lim\limits_{x\to 3^+}\dfrac{3}{\sqrt{2x}}\cdot\dfrac{x-3}{\sqrt{x-3}}\cdot\dfrac{1}{\sqrt{3x}+3}=$

(using $\dfrac{a}{\sqrt{a}}=\sqrt{a}$ for $a>0$)

$=\lim\limits_{x\to 3^+}\dfrac{3}{\sqrt{2x}}\cdot\sqrt{x-3}\cdot\dfrac{1}{\sqrt{3x}+3}=\frac{3}{\sqrt{6}}\cdot 0\cdot\frac{1}{6}=0.$

For 5b) as $x \to +\infty$,
the type is "$\frac{\infty}{\infty}$" so we have to divide by the highest power in the denominator in the original from by $x = \sqrt{x^2}$ $(x > 0)$:

$$\lim_{x\to+\infty} \frac{\sqrt{3x} - 3}{\sqrt{2x^2 - 6x}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to+\infty} \frac{\sqrt{\frac{3}{x}} - \frac{3}{x}}{\sqrt{2 - \frac{6}{x}}} = \frac{0-0}{\sqrt{2}} = 0$$

OR we can used the simplified expresion from a)

$$\lim_{x\to+\infty} \frac{3}{\sqrt{2x}} \cdot \sqrt{x-3} \cdot \frac{1}{\sqrt{3x}+3} = \lim_{x\to+\infty} 3\sqrt{\frac{x-3}{2x}} \cdot \frac{1}{\sqrt{3x}+3} =$$

$$= \lim_{x\to+\infty} 3\sqrt{\frac{1}{2} - \frac{3}{2x}} \cdot \lim_{x\to+\infty} \frac{1}{\sqrt{3x}+3} = \frac{3}{\sqrt{2}} \cdot 0 = 0 \text{ since } "\frac{1}{\infty}" = 0.$$

For 5c) as $x \to 0$
the limit DNE (does not exist neither as a number nor as $\pm\infty$)
since the function is not defined for small negative x $(\sqrt{neg})$

For 6)

For $f(x) = \dfrac{\sqrt{3-x}}{x^2 - 4x + 3}$

For 6 a) as $x \to 3^-$
the type is "$\frac{0}{0}$" and the function is defined for $x < 3$ and $x \neq 1$ we can simplify

$$f(x) = \frac{\sqrt{3-x}}{x^2 - 4x + 3} = \frac{\sqrt{3-x}}{(x-3)(x-1)} = \frac{\sqrt{3-x}}{-(3-x)(x-1)} = \frac{\sqrt{3-x}}{-\sqrt{3-x}\sqrt{3-x}(x-1)} = -\frac{1}{\sqrt{3-x}\,(x-1)}$$

Now the type is "$\frac{-1}{0^+\cdot(2)}$" = "$\frac{1}{0^{--}}$" and the limit is $-\infty$.

For 6b) as $x \to 1^+$
We can use the simplification from above or at least identify the type "$\frac{\sqrt{2}}{0}$"
so we have to investigate the sign of the bottom
Since $x > 1$ and $f(x) = \dfrac{\sqrt{3-x}}{(x-3)(x-1)}$ we can see that the type is :"$\frac{\sqrt{2}}{(-2)\cdot 0^+}$" = "$\frac{1}{0^-}$"
and the limit is $-\infty$.

For 6c)
as $x \to +\infty$.
the limit DNE (does not exists) since the function is not defined for big positive x .

For 7)
For $g(x) = \sqrt{3+x}$ and $f(x) = \sqrt{x-5}$
fiind the domains $D_g = [-3, +\infty)$ since it must $3 + x \geq 0$ and
$D_f = [5, +\infty)$ since it must $x - 5 \geq 0$
Now
$f \circ g(x) = f(g(x)) = \sqrt{(..) - 5} = \sqrt{\sqrt{3+x} - 5}$
we must start in D_g i.e. $x \in [-3, +\infty)$ and we have to guarantee that
$\sqrt{3+x} - 5 \geq 0$ solve: $\sqrt{3+x} \geq 5$
both sides are positive so we can square $(3 + x) \geq 25$, and $x \geq 22$,together
$D_{f\circ g} = [22, +\infty)$
then
$g \circ g(x) = \sqrt{3 + (..)} = \sqrt{3 + \sqrt{3+x}}$
we must start in $D_g = [-3, +\infty)$ and quarantee that $3 + \sqrt{3+x} \geq 0$

but it is always true for any $x \in [-3, +\infty)$ so $D_{g\circ g} = [-3, +\infty)$.

For 8a) as $x \to 0$

the type is "$\frac{0}{0}$" and

if x is a small # ,neg. or pos,$x - 3$ is close to -3 so negative

thus $|x-3| = -(x-3) = 3 - x$

and $x + 3$ is close to 3 so $x + 3$ is positive thus $|x+3| = x + 3$

therefore

$$f(x) = \frac{|x-3| - |x+3|}{x} = \frac{3 - x - (x+3)}{x} = \frac{-2x}{x} = -2 \text{ for } x \neq 0$$

so the limit is $L = -2$.

ALSO

$$\frac{|x-3| - |x+3|}{x} \cdot \frac{|x-3| + |x+3|}{|x-3| + |x+3|} = \frac{|x-3|^2 - |x+3|^2}{x \cdot (|x-3| + |x+3|)} = \frac{(x-3)^2 - (x+3)^2}{x \cdot (|x-3| + |x+3|)} =$$

$$= \frac{x^2 - 6x + 3^2 - (x^2 + 6x + 3^2)}{x \cdot (|x-3| + |x+3|)} = \frac{-12x}{x \cdot (|x-3| + |x+3|)} = \frac{-12}{(|x-3| + |x+3|)}$$

for any $x \neq 0$

Now the limit as $x \to 0$ is $L = \frac{-12}{3+3} = -2$.

For8b) as $x \to -\infty$

For negative number x with big magnitude $x-3$ is negative thus $|x-3| = -(x-3) = 3-x$,
also $x + 3$ is negative thus $|x+3| = -(x+3) = -3 - x$ then

$$f(x) = \frac{|x-3| - |x+3|}{x} = \frac{3 - x + x + 3}{x} = \frac{6}{x}$$ so the type of the limit is "$\frac{1}{\infty}$" and $L = 0$.

ALSO for both b) and c)

using the siplification from above $f(x) = \dfrac{-12}{(|x-3| + |x+3|)}$ so the type is "$\frac{-12}{\infty}$"
and the limit is 0.

For 8 c) as $x \to +\infty$.

For x big positive both expressions $x - 3$ and $x + 3$ are positive so we can ignore absolute values and

$f(x) = \dfrac{x - 3 - (x+3)}{x} = \frac{-6}{x}$ and the type of the limit is "$\frac{-6}{\infty}$" and the limit is 0.

For 9)

For $g(x) = \sqrt{3-x}$ and $f(x) = \dfrac{6}{3x-1}$

find the domains $D_f = \left\{x \neq \frac{1}{3}\right\}$ since it must $3x - 1 \neq 0$;

$D_g = (-\infty, 3]$ since it must $3 - x \geq 0$.

Now

$$f \circ f(x) = f(f(x)) = \frac{6}{3(...) - 1} = \frac{6}{3 \cdot \frac{6}{3x-1} - 1} = \frac{6}{\frac{18-(3x-1)}{3x-1}} = \frac{6\,(3x-1)}{19 - 3x}$$

we must start in D_f i.e. $x \neq \frac{1}{3}$ and we have to guarantee that $19 - 3x \neq 0$ so $x \neq \frac{19}{3}$

together $D_{f\circ f} = \left\{x \neq \frac{1}{3}, \frac{19}{3}\right\}$

then

$$g \circ f(x) = \sqrt{3 - (..)} = \sqrt{3 - \tfrac{6}{3x-1}} = \sqrt{\tfrac{3(3x-1)-6}{3x-1}} = \sqrt{\tfrac{9x-9}{3x-1}} = 3\sqrt{\tfrac{x-1}{3x-1}}$$

we must start in $D_f : x \neq \frac{1}{3}$ and quarantee that $\dfrac{x-1}{3x-1} \geq 0$,

split points are $x = 1, \frac{1}{3}$ testing: $-^{pos} - -_{\frac{1}{3}} - -^{neg} - -_{1} - -^{pos}-$

so the domain is $D_{g\circ f} = \left(-\infty, \frac{1}{3}\right) \cup [1, +\infty)$