MATH 249 Worksheet #5

1. How much money one has to invest today at the interest of 3% compounded quaterly to get \$10,000 in 10 years?

2. Find the domain and derivative of $f(x) = (3x)^{\pi} + \pi^{3x} + (\pi x)^{3x}$.

3. Find y' if $y = 2^{x^4} + \frac{2}{x^4} + \left(\frac{1}{x}\right)^x$,for $x > 0$.

4. How long does it take to double your investment if the interest of 7 % is compounded
   (a) yearly? (b) monthly?

5. In the first few weeks after birth,a baby gains weight at a rate proportional to its weight.A baby weighing 4kg at birth weighs 4.4kg after 2 weeks. How much did it weigh 4 days after birth ?

6. For $f(x) = 3^x \ln \frac{3}{x}$ find $f'(3)$.

7. Find y' if $y = x^{x^2} + \ln \frac{1}{1-x}$,for $0 < x < 1$.

8. After 3 days a sample of radon-222 decayed to 58% of its original amount.What is half-life of radon-222?

9. Evaluate (a) $\lim\limits_{x\to+\infty} \dfrac{x}{2^x - 1}$ (b) $\lim\limits_{x\to-\infty} \dfrac{x}{2^x - 1}$ (c) $\lim\limits_{x\to 0} \dfrac{x}{2^x - 1}$

10. Evaluate
    (a) $\lim\limits_{x\to\infty} \dfrac{x^2}{e^{3x}}$ (b) $\lim\limits_{x\to-\infty} \dfrac{x^2}{e^{3x}}$ (c) $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{x}$ (d) $\lim\limits_{x\to 0^+} \dfrac{(\ln x)^2}{x}$.

11. Solve for x : $\dfrac{1}{3^{x+1}} = \dfrac{5}{4^x}$.

    .

12. Solve for x: (a) $\frac{1}{2}\ln(x+3) + 1 = 0$; (b) $3^{x^2} = 9^{x-3}$.

SOLUTION **For1)**

the formula to use is $A(t) = A_0\left(1 + \dfrac{p}{100n}\right)^{nt}$ where $t = 10, n = 4, p = 3$

$A = 10,000$ $A_0 = ?$ is the initial amount to invest so

$10000 = A_0(1 + \dfrac{3}{400})^{40} = A_0(\dfrac{403}{400})^{40}$ multiply by reciprocal to isolate A_0

$A_0 = 10000(\dfrac{400}{403})^{40} = \$7416.48$

For 2)

for $x>0$ $\quad f(x)=(3x)^{\pi}+\pi^{3x}+(\pi x)^{3x}=3^{\pi}x^{\pi}+\left(\pi^3\right)^x+e^{3x\ln\pi x}$

power + exp.f + e^u so by Chain Rule $(e^u)'=e^u u'$

$f'(x)=3^{\pi}\left(x^{\pi}\right)'+\left[\left(\pi^3\right)^x\right]'+e^{3x\ln\pi x}\left[3x\ln(\pi x)\right]'$ (Pr.R.)

$=\pi 3^{\pi}x^{\pi-1}+\left(\pi^3\right)^x\ln\pi^3+e^{3x\ln\pi x}\left[3\ln\pi x+3x\cdot\dfrac{1}{\pi x}\cdot\pi\right]=$

$=\pi 3^{\pi}x^{\pi-1}+3\left(\pi^3\right)^x\ln\pi+e^{3x\ln\pi x}\left[3\ln\pi x+3\right]$

ALSO

change all terms into e^u and then Chain rule

$f(x)=(3x)^{\pi}+\pi^{3x}+(\pi x)^{3x}=e^{\pi\ln 3x}+e^{3x\ln\pi}+e^{3x\ln\pi x}$

so

$f'(x)=e^{\pi\ln 3x}\left(\pi\ln 3x\right)'+e^{3x\ln\pi}\left(3x\ln\pi\right)'+e^{3x\ln\pi x}\left(3x\ln\pi x\right)'=$

$=e^{\pi\ln 3x}\left(\pi\cdot\dfrac{3}{3x}\right)+e^{3x\ln\pi}\left(3\ln\pi\right)+e^{3x\ln\pi x}\left[3\ln\pi x+3x\cdot\dfrac{1}{\pi x}\cdot\pi\right]=$

$=\dfrac{\pi}{x}(3x)^{\pi}+\left(3\ln\pi\right)\pi^{3x}+\left[3\ln\pi x+3\right](\pi x)^{3x}$

For 3)

$y'=\left(e^{x^4\ln 2}\right)'+\left(2x^{-4}\right)'+\left(e^{x\ln\frac{1}{x}}\right)'=e^{x^4\ln 2}\left(x^4\ln 2\right)'-8x^{-5}+e^{-x\ln x}\cdot\left(-x\ln x\right)'=$

$=4x^3\ln 2\,e^{x^4\ln 2}-8x^{-5}+e^{-x\ln x}\left(-1\cdot\ln x-x\cdot\frac{1}{x}\right)=4x^3\ln 2\,e^{x^4\ln 2}-8x^{-5}-e^{-x\ln x}\left(\ln x+1\right)$

OR

by log.diff.BUT only for the last term $\quad u=\left(\frac{1}{x}\right)^x=e^{x\ln\frac{1}{x}}$ so

$\ln u=x\ln\frac{1}{x}=x\ln x^{-1}=-x\ln x$

and

$\frac{1}{u}\cdot u'=-\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)=-(\ln x+1)$ so $u'=-\left(\frac{1}{x}\right)^x(\ln x+1)$

for the first term we can use $(2^u)'=2^u\cdot\ln 2\cdot u'$ so $\left(2^{x^4}\right)'=2^{x^4}\cdot\ln 2\cdot 4x^3$

and the second term is just a power so $\left(2x^{-4}\right)'=-8x^{-5}$.

Together

$y'=4x^3\ln 2\cdot 2^{x^4}-8x^{-5}-\left(\frac{1}{x}\right)^x(\ln x+1)$ $\quad$ as above.

For 4a)

we have to solve for t $\quad 2A_0=A_0\left(1+\frac{7}{100}\right)^t$ cancel A_0

thus $\quad 2=(1.07)^t$ $\quad$ take ln of both sides:

$\ln 2=t\ln 1.07$, so $t=\frac{\ln 2}{\ln 1.07}=10.24$

We need 10 years and almost 3 months.

For 4b)

we have to solve for t $\quad 2A_0=A_0\left(1+\frac{7}{1200}\right)^{12t}$

as above

thus $2=\left(\frac{1207}{1200}\right)^{12t}$ and

$\ln 2=12t\ln\dfrac{1207}{1200}$ $\quad t=\frac{\ln 2}{12(0.0058163)}=9.929.$

So this time we need less than 10 years.

For 5)

for the weight after t **weeks** $W(t)=4e^{kt}$, given $\quad W(2)=4.4=4e^{2k}$

solve for k :

$\frac{4.4}{4} = e^{2k}$, apply ln to both sides: $\ln 1.1 = 2k$, so $k = \frac{\ln 1.1}{2}$
Now,after 4 days $= \frac{4}{7}$ of a week ,
so

$$W = 4e^{k\frac{4}{7}} = 4e^{\frac{2}{7}\cdot \ln 1.1} = 4.11 \text{ kg}.$$

OR

$W(t) = 4e^{kt}$ where we measure time t **in days**
then 2 weeks is 14 days

$$W(14) = 4.4 = 4e^{14k} \text{ then } k = \frac{\ln 1.1}{14}$$

and for $t = 4$ $\quad W = 4e^{k4} = 4e^{4\frac{\ln 1.1}{14}} =$ (as above)$= 4.11$ kg

For 6)

first simplify $f(x) = 3^x \cdot (\ln 3 - \ln x)$, then use Product Rule

$f'(x) = (3^x)'(\ln 3 - \ln x) + 3^x (\ln 3 - \ln x)' = 3^x \ln 3 (\ln 3 - \ln x) + 3^x \left(\frac{-1}{x}\right)$

since $(\ln 3)' = 0$ then substitute $x = 3$

and $\quad f'(3) = 3^3 \ln 3 (\ln 3 - \ln 3) + 3^3 \left(\frac{-1}{3}\right) = -9.$

For 7)

simplify first

$y = x^{x^2} + \ln \frac{1}{1-x} = e^{x^2 \ln x} - \ln(1-x) \qquad y' = e^{x^2 \ln x}\left(x^2 \ln x\right)' - \frac{1}{1-x}(1-x)' =$

$= e^{x^2 \ln x}(2x \ln x + x) + \frac{1}{1-x}$

OR you can use log.diff. but only for the first part

$u = x^{x^2} \qquad \ln u = x^2 \ln x \qquad \frac{u'}{u} = 2x \ln x + x^2 \cdot \frac{1}{x} \qquad u' = x^{x^2}(2x \ln x + x)$

and $\quad y' = u' + \frac{1}{1-x}.$

For 8)

the correct formula $\quad A(t) = A_0 e^{kt}$ where $k < 0, t$ in **days**, $A_0 = 100\%$

first info $\quad$ if $t = 3 \quad 58 = 100e^{3k}$ solve for $k \quad \ln \frac{58}{100} = 3k$

$k = \frac{\ln 0.58}{3} = -0.1815757$

so $\quad A(t) = 100e^{kt}$ for k calculated above ;

now half-life T $\quad$ means we got 50%

$50 = 100e^{kT} \quad$ where $k = \frac{\ln 0.58}{3} \qquad \frac{50}{100} = e^{kT}$

solve for $T \qquad \ln 0.5 = kT \qquad T = \frac{3 \ln 0.5}{\ln 0.58} = 3.8174$ days

For 9) $\quad$ **for a)**

$\lim_{x\to+\infty} \frac{x}{2^x - 1}$ $\quad$ the type is "$\frac{+\infty}{+\infty}$" so we can use L'Hop.rule

$\lim_{x\to+\infty} \frac{(x)'}{(2^x-1)'} = \lim_{x\to+\infty} \frac{1}{2^x \ln 2} =$ "$\frac{1}{\infty}$" $= 0$

for b)

$\lim_{x\to-\infty} \frac{x}{2^x - 1}$ since " $2^{-\infty}$" $= 0$ No L'H.R. $\qquad \lim_{x\to-\infty} \frac{x}{2^x - 1} = \frac{-\infty}{-1} = +\infty.$

for c)

$\lim_{x\to 0} \frac{x}{2^x - 1}$ the type is "$\frac{0}{0}$" so L'Hop.Rule again $\quad \lim_{x\to 0} \frac{x}{2^x-1} = \lim_{x\to 0} \frac{1}{2^x \ln 2} = \frac{1}{\ln 2}$

For 10)

for a)

the type is "$\frac{\infty}{\infty}$" so use L'Hop.Rule

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}} = \lim_{x\to\infty} \frac{2x}{3e^{3x}} \text{ (again)} = \lim_{x\to\infty} \frac{2}{9e^{3x}} = \text{"}\frac{2}{\infty}\text{"} = 0$$

for b)

the type is "$\frac{\infty}{0^+}$" since "$e^{-\infty}$" $= 0$

so $\lim_{x\to-\infty} \frac{x^2}{e^{3x}} = +\infty\,(+\infty) = +\infty$ since "$\frac{1}{0^+}$" $= +\infty$. No L'H.R.

also $\lim_{x\to-\infty} \frac{x^2}{e^{3x}} = \lim_{x\to-\infty} x^2 e^{-3x} = +\infty \cdot e^{+\infty} = +\infty$

for c) $\quad \lim_{x\to\infty} \frac{(\ln x)^2}{x} = \text{"}\frac{\infty}{\infty}\text{"}$ (L'H.R)$= \lim_{x\to\infty} \frac{2\,(\ln x)\,\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x} =$

again L'H.R. $\quad = \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \text{"}\frac{2}{\infty}\text{"} = 0$

for d) $\quad \lim_{x\to 0^+} \frac{(\ln x)^2}{x} = \text{"}\frac{\infty}{0^+}\text{"} = \lim_{x\to 0^+} (\ln x)^2 \cdot \frac{1}{x} = (-\infty)^2\,(+\infty) = +\infty$

(No L'H.R.)

For 11)

cross multiply first,so $4^x = 5\cdot 3^{x+1}$, then apply ln to both sides

$\ln 4^x = \ln(5\cdot 3^{x+1}) = \ln 5 + \ln 3^{x+1}$

thus $\quad x\ln 4 = \ln 5 + (x+1)\ln 3 \qquad x\ln 4 = \ln 5 + x\ln 3 + \ln 3$

and $x\ln 4 - x\ln 3 = \ln 5 + \ln 3$

So $x(\ln 4 - \ln 3) = \ln(5\cdot 3)$ and finally $\quad x\ln\frac{4}{3} = \ln 15$

$$x = \frac{\ln 15}{\ln\frac{4}{3}} = \frac{\ln 15}{\ln 4 - \ln 3}.$$

For 12a)

$\frac{1}{2}\ln(x+3) = -1 \qquad \ln(x+3) = -2$

then exp.f. to both sides and $\quad e^{\ln(x+3)} = e^{-2}$

$(x+3) = e^{-2}$

and so $x = e^{-2} - 3$

OR $\quad \frac{1}{2}\ln(x+3) = \ln\sqrt{x+3} = -1$

then apply exp.f to get $\quad \sqrt{x+3} = e^{-1}$

then square to get the above.

b)

Take log of both sides: $\quad \ln 3^{x^2} = \ln 9^{x-3}$

$x^2\ln 3 = (x-3)\ln 9 = (x-3)\ln 3^2 = (x-3)\cdot 2\ln 3$

cancel $\ln 3$ and $\quad x^2 = 2(x-3) = 2x - 6$

everything on one side :$x^2 - 2x + 6 = 0$, discriminant of this polynomial is

$D = (-2)^2 - 4\cdot 1\cdot 6 = -20$ so no real roots exist and the problem has **NO solution**.

Also we can change both sides to the same base: $3^{x^2} = (3^2)^{x-3} = 3^{2x-6}$

and by comparing the exponents we get the same quadratic equation.