MATH 251/249
Midterm Handout

For 1)

a) $\lim\limits_{x\to\infty}\left(x^2 - x^2\cos\frac{1}{x}\right) = \lim\limits_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right)$ "$\infty\cdot 0$" $\quad u = \frac{1}{x}, u \to 0^+$

so

$$\lim_{x\to\infty}... = \lim_{u\to 0^+}\frac{(1-\cos u)}{u^2}\cdot\frac{(1+\cos u)}{(1+\cos u)} = \lim_{u\to 0^+}\frac{(1-\cos^2 u)}{u^2(1+\cos u)} =$$

$$= \lim_{u\to 0^+}\frac{\sin^2 u}{u^2(1+\cos u)} = \lim_{u\to 0^+}\left(\frac{\sin u}{u}\right)^2\cdot\lim_{u\to 0^+}\frac{1}{(1+\cos u)} = 1\cdot\frac{1}{2} = \frac{1}{2}..$$

b) $\lim\limits_{x\to 0}\left(x^2 - x^2\cos\frac{1}{x}\right) = \lim\limits_{x\to 0} x^2\left(1-\cos\frac{1}{x}\right)$ "$0\cdot DNE$" $= 0$ by Sq,Th

since

$0 \le 1-\cos\theta \le 2$ for any $\theta \qquad 0 \le x^2\left(1-\cos\frac{1}{x}\right) \le 2x^2 \to 0$ as $x\to 0$.

For 2)

a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x-\pi} = \dfrac{0}{-\pi} = 0$

b) $\lim\limits_{x\to\pi}\dfrac{\sin x}{x-\pi} =$ "$\dfrac{0}{0}$" $= \lim\limits_{u\to 0}\dfrac{\sin(u+\pi)}{u} = \lim\limits_{u\to 0}\dfrac{\sin u\cos\pi + \cos u\sin\pi}{u} = \lim\limits_{u\to 0} -\dfrac{\sin u}{u} = -1$

where $x-\pi = u, x = u+\pi$ and $\cos\pi = -1$,$\sin\pi = 0$

c) $\lim\limits_{x\to-\infty}\dfrac{\sin x}{x-\pi} = 0$ by Squeeze Th. since $-1 \le \sin x \le 1$ and $\quad x-\pi < 0$

$$\frac{-1}{x-\pi} \ge \frac{\sin x}{x-\pi} \ge \frac{1}{x-\pi} \text{ and } \lim_{x\to-\infty}\frac{\pm 1}{x-\pi} = 0.$$

For 3)

if $x = \dfrac{-1}{2}$ then $y = \dfrac{\cos\left(\frac{-\pi}{2}\right)}{\frac{3}{2}} = 0$ so $\qquad y = m\left(x+\dfrac{1}{2}\right)$

for m use Q.R. to find $y' = \left(\dfrac{\cos\pi x}{1-x}\right)' = \dfrac{-\pi\sin\pi x\cdot(1-x) - \cos\pi x\cdot(-1)}{(1-x)^2}$

at $x = -\frac{1}{2} \qquad \cos\frac{-\pi}{2} = 0, \sin\frac{-\pi}{2} = -1$ so

$$m = \frac{-\pi(-1)\cdot\frac{3}{2} - 0}{\left(\frac{3}{2}\right)^2} = \frac{\pi}{\frac{3}{2}} = \frac{2\pi}{3}\text{... slope} \qquad \text{tangent } y = \frac{2\pi}{3}\left(x+\frac{1}{2}\right).$$

For 4 a)

for $y = \left(\sin\frac{1}{\sqrt{x^4+1}}\right)^3$ use Chain Rule twice

$$y' = 3\left(\sin\tfrac{1}{\sqrt{x^4+1}}\right)^2\left(\sin\tfrac{1}{\sqrt{x^4+1}}\right)' = 3\left(\sin\tfrac{1}{\sqrt{x^4+1}}\right)^2\left(\cos\tfrac{1}{\sqrt{x^4+1}}\right)\left((x^4+1)^{-\frac{1}{2}}\right)' =$$

$$= 3\left(\sin\tfrac{1}{\sqrt{x^4+1}}\right)^2\left(\cos\tfrac{1}{\sqrt{x^4+1}}\right)\left(-\tfrac{1}{2}\right)\left((x^4+1)^{-\frac{3}{2}}\right)4x^3 = \frac{-6x^3\sin^2\frac{1}{\sqrt{x^4+1}}\cos\frac{1}{\sqrt{x^4+1}}}{(x^4+1)^{\frac{3}{2}}}$$

For 4 b)

For $y = \cos(3-2x^2)$ first the first derivative by Chain Rule :

$y' = -\sin(3-2x^2)\cdot(3-2x^2)' = .-\sin(3-2x^2)\cdot(-4x) = 4x\sin(3-2x^2)$

then the second derivative by Product and Chain Rules

$y' = 4\left[x\sin(3-2x^2)\right]' = 4\left[(x)'\sin(3-2x^2) + x\left(\sin(3-2x^2)\right)'\right] =$
$= 4\left[1\cdot\sin(3-2x^2) + x\cos(3-2x^2)(3-2x^2)'\right] =$
$= 4[\sin(3-2x^2) - 4x^2\cos(3-2x^2)]$

For 5)

the function $f(x) = x - 2\sin(\pi x)$ is continuous everywhere
and
$f(\frac{1}{2}) = \frac{1}{2} - 2\sin\frac{\pi}{2} = -\frac{3}{2} < 0$ and $f(1) = 1 - 2\sin\pi = 1 > 0$
so by IVT there must be an c between $\frac{1}{2}$ and 1 where $f(c) = 0$

For 6)

the polynoial $p(x) = 2x^3 - 6x^2 + 7$ is continuous everywhere
we can have 3 roots or at least one;to decide find Critical points first
$p'(x) = 6x^2 - 12x = 6x(x-2) = 0 \qquad x = 0, 2$
then find y-values $\qquad x = 0 \to y = 7$
$x = 2 \to y = 2\cdot 2^3 - 6\cdot 2^2 + 7 = 2^3(2-3) + 7 = -1 < 0j$
so we have 3 roots
to locate at least one test some values of $p:\quad p(0) = 7 > 0$
and $p(-1) = -2 - 6 + 7 = -1 < 0$ so by IVT(intermediate value theorem
there must be one root r_1 between -1 and 0 $\quad r_1 \in (-1, 0)$
since $p(1) = 2 - 6 + 7 = 3 > 0$ and $p(2) = 16 - 24 + 7 = -1 < 0$
so by IVT there is another root between 1 and 2 $\quad r_2 \in (1, 2)$
finally $p(3) = 54 - 54 + 7 = 7 > 0$
by IVT there is a root $\quad r_3 \in (2, 3)$

For 7)

$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-\frac{1}{25}} = \pm\sqrt{\frac{24}{25}} = \pm\frac{\sqrt{24}}{5}$
but since θ is in the second quadrant cos must be negative
$\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{\sqrt{24}}$

For 8)

$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\frac{4}{9}} = \pm\sqrt{\frac{5}{9}} = \pm\frac{\sqrt{5}}{3}$
but since θ is in the forth quadrant sin must be negative
$\sin\theta = -\dfrac{\sqrt{5}}{3}$. Now $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(\frac{-\sqrt{5}}{3}\right)\cdot\frac{2}{3} = -\dfrac{4\sqrt{5}}{9}$.

For 9)

$$f(x) = \begin{cases} \left(\frac{2}{2x+1} - 3\right)(4x^2-1) & \text{for} \quad x < -\frac{1}{2} \\ ax + b & \text{for} \quad -\frac{1}{2} \le x \le 2 \\ \cos(-\frac{\pi}{x}) & \text{for} \quad x > 2 \end{cases} .$$

the function is continuous except $x = -\frac{1}{2}$ and $x = 2$
$f(-\frac{1}{2}) = \frac{-1}{2}a + b = \lim\limits_{x\to-\frac{1}{2}^+} f(x)$

$$\lim_{x\to-\frac{1}{2}^-} f(x) = \lim_{x\to-\frac{1}{2}^-}\left(\frac{2}{2x+1} - 3\right)(4x^2-1) = \lim_{x\to-\frac{1}{2}^-}\left(\frac{2-3(2x+1)}{2x+1}\right)(4x^2-1)$$
$$= \lim_{x\to-\frac{1}{2}^-}\frac{2-6x-3}{2x+1}\cdot(2x-1)(2x+1) = \lim_{x\to-\frac{1}{2}^-}(-1-6x)(2x-1) = 2\cdot(-2) = -4$$

All 3 numbers must be the same so $\quad \frac{-1}{2}a + b = -4$
Similarly,for $x = 2$
$f(2) = 2a + b = \lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+}\cos\frac{-\pi}{x} = \cos\frac{-\pi}{2} = 0$

All 3 numbers must be the same so $2a+b=0 \longrightarrow\longrightarrow b=-2a$
back to $\frac{-1}{2}a+b=-4$ and.$a\left(\frac{-1}{2}-2\right)=-4 \Rightarrow a\left(\frac{-5}{2}\right)=-4$
so $a=(-4)\left(\dfrac{-2}{5}\right)=\dfrac{8}{5}$ and $b=-2a=\dfrac{-16}{5}$.

For 10)

$$f(x)=\begin{cases} \cos(\pi x)-2\sin\frac{\pi x}{2} & \text{for} \quad x>3 \\ ax^2+b & \text{for} \quad 0\le x\le 3 \\ 6\cdot\dfrac{\sqrt{9-x}-3}{x} & \text{for} \quad x<0 \end{cases}$$

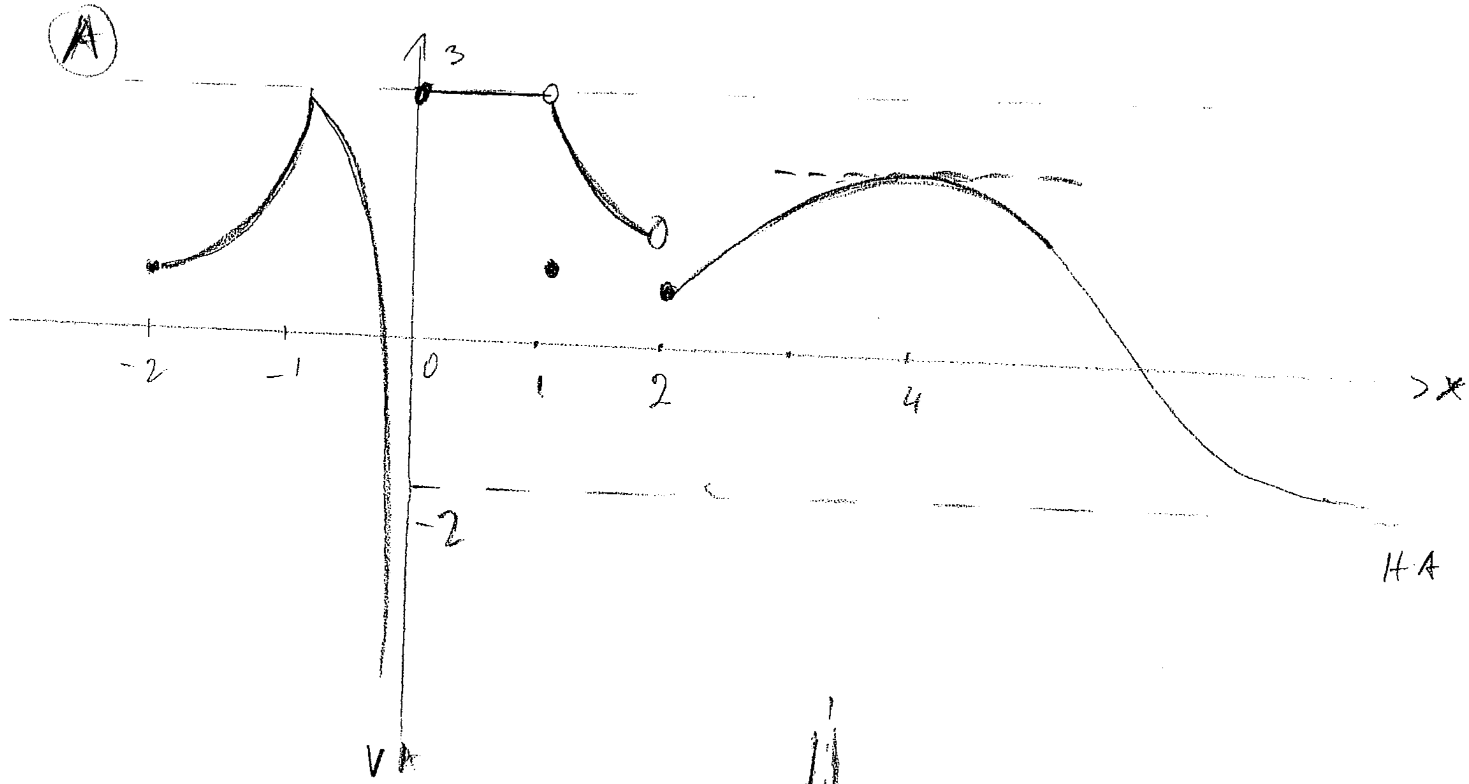
the function is continuous for any x except $x=3$ and $x=0$
$f(3)=9a+b=\lim\limits_{x\to 3^-} f(x)$ and
$\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}\left[\cos(\pi x)-2\sin\frac{\pi x}{2}\right]=\cos(3\pi)-2\sin\frac{3\pi}{2}=-1-2(-1)=1$
All 3 numbers must be the same so $9a+b=1$
Similarly,for $x=0$

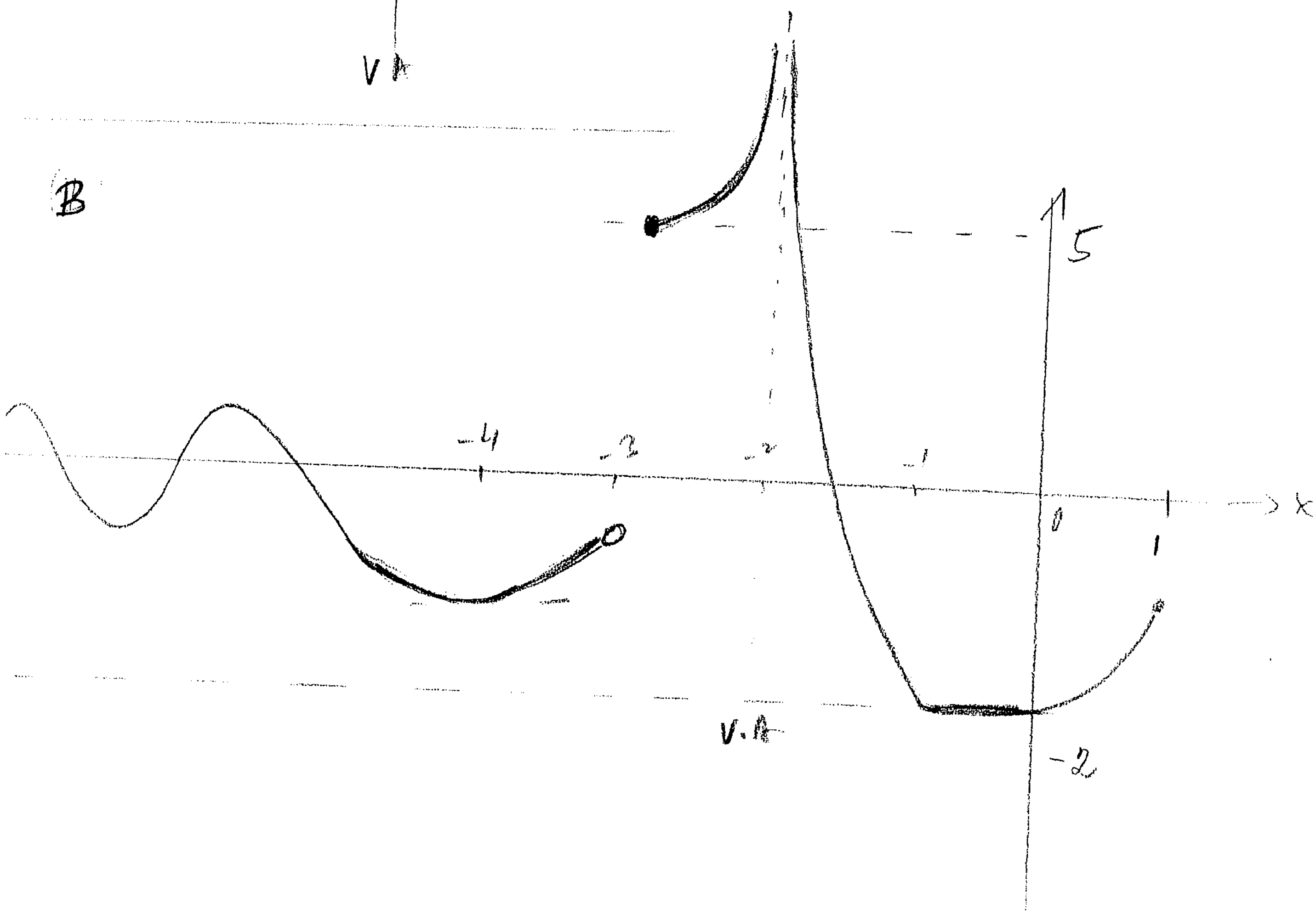
$$f(0)=b=\lim_{x\to 0^+} f(x) \text{ and } \lim_{x\to 0^-} 6\cdot\frac{\sqrt{9-x}-3}{x}\cdot\frac{\sqrt{9-x}+3}{\sqrt{9-x}+3}=6\lim_{x\to 0^-}\frac{9-x-3^2}{x\left(\sqrt{9-x}+3\right)}=$$

$$=6\lim_{x\to 0^-}\frac{-x}{x\left(\sqrt{9-x}+3\right)}=6\lim_{x\to 0^-}\frac{-1}{(\sqrt{9-x}+3)}=\frac{-6}{6}=-1$$

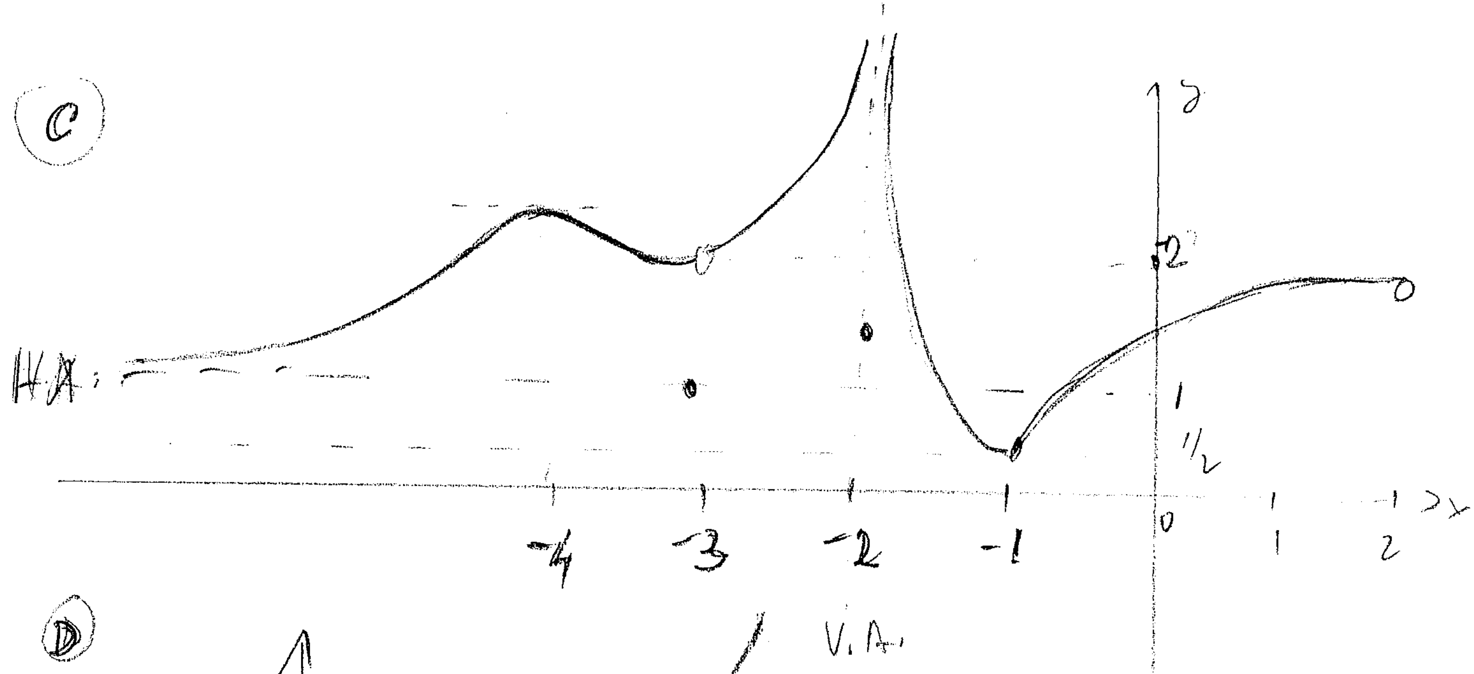
All 3 numbers must be the same so $b=-1$
substitute $b=-1$ in the first equation $9a-1=1 \quad a=\frac{2}{9}$ and $b=-1$.

For 11)

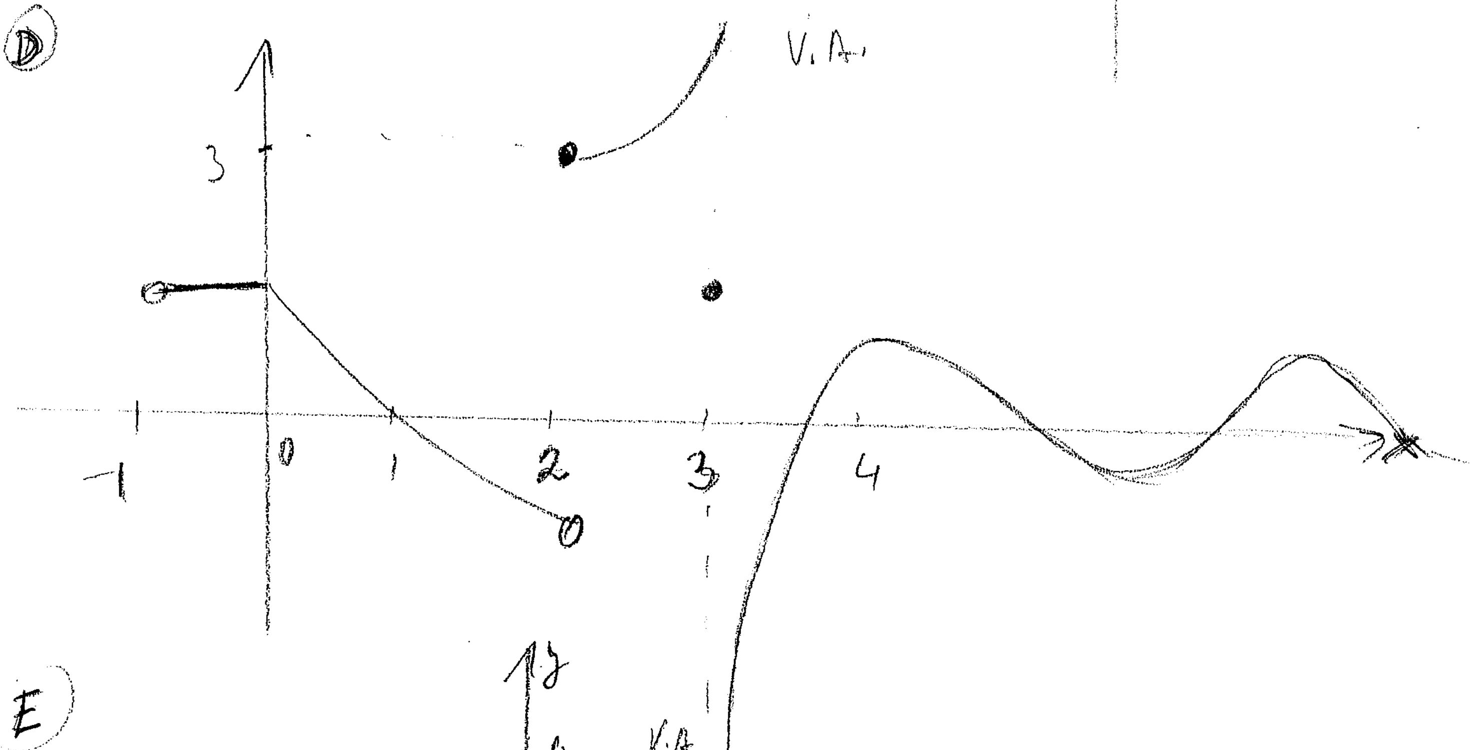

A
3
-2
-1
0
1
2
4
x
-2
HA
VA


B
5
-4
-3
-2
-1
0
1
x
-2
V.A

C



D



E