NAME: ______________________

The University of Calgary
Department of Mathematics and Statistics

MATHEMATICS 251 - L08
Midterm Examination
Fall 2002

Thursday, October 31, 2002 Time: Approximately 1 hour

SHOW YOUR WORK

TOTAL POINTS = 40

NO CALCULATORS OR NOTES ARE PERMITTED

I.D. ______________________

[8] 1. In each of (a), (b), (c), find $\dfrac{dy}{dx}$.

(a) $y = 2^{\tan(x^2)}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan(x^2)\right) 2^{\tan(x^2)} \ln 2 = 2x \sec^2(x^2) \ln 2 \; 2^{\tan(x^2)}$$

OR

$$\ln y = \tan(x^2) \ln 2, \quad \frac{1}{y}\frac{dy}{dx} = \sec^2(x^2)(2x) \ln 2$$

$$\frac{dy}{dx} = 2^{\tan(x^2)} \sec^2(x^2)\, 2x \ln 2$$

3

(b) $y = \dfrac{1+\sin x}{1+\cos x}$.

$$\frac{dy}{dx} = \frac{(\cos x)(1+\cos x) - (1+\sin x)(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{1 + \sin x + \cos x}{(1+\cos x)^2}$$

3

(c) $y = 4x^{\frac{3}{2}}(1+x)^3$.

$$\frac{dy}{dx} = 4\left[\frac{3}{2} x^{1/2} (1+x)^3 + x^{3/2}(3)(1+x)^2\right]$$

$$= 6x^{1/2}(1+x)^3 + 12x^{3/2}(1+x)^2$$

2

[4] 2. Show that the equation $x^3 = 2x + 1$ has at least one solution in $[0, 2]$.

$f(x) = x^3 - 2x - 1$

$f(0) = -1 < 0$

$f(2) = 8 - 4 - 1 = 3 > 0$

Since f is continuous, then, by the IMV, f has at least one root in $[0, 2]$

[5] 3.

(a) State the local linear approximation of $f(x) = \sqrt{4 + x}$ at $x_0 = 0$.

$f(x) \doteq f(x_0) + (x - x_0) f'(x_0)$ (1)

$f(x) = \sqrt{4+x}$, $x_0 = 0$, $f(x_0) = 2$, $f'(x) = \frac{1}{2\sqrt{4+x}}$, $f'(x_0) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$\therefore \sqrt{4+x} \doteq 2 + \frac{x}{4}$ (2)

(b) Use your answer in (a) to approximate $\sqrt{4.4}$

$\sqrt{4.4} = \sqrt{4+x}$, $4.4 = 4 + x$, $x = .4$ (2)

$\sqrt{4.4} \doteq 2 + \frac{.4}{4} = 2.1$

[5] 4. Find $\frac{dy}{dx}$ at the point (2,-1) on the curve $x^2 + xy + 2y^3 = \cancel{4}\ 0$

$$\frac{d}{dx}(x^2 + xy + 2y^3) = \frac{d}{dx}(0) = 0$$

$$2x + y + x\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x + 6y^2) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x+6y^2} = \frac{(-2)(2)+1}{2+6} = -\frac{3}{8}$$

[2] 5. Let $f(x) = x^4 + 3x + 1$. Find $\frac{d}{dx}(f^{-1}(x))$.

Let $y = f^{-1}(x)$. Then $f(y) = x$

$$\therefore\ x = y^4 + 3y + 1$$

$$\frac{dx}{dy} = 4y^3 + 3$$

$$\frac{dy}{dx} = \frac{1}{4y^3+3}$$

[3] 6. If x and y are functions of t with $\frac{dx}{dt} = -2$, find $\frac{dy}{dt}$ when $x = 4$ given that $x^2 + y^2 = 25$.

$$\frac{d}{dt}\left(x^2+y^2\right) = \frac{d}{dt}(25) = 0$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0, \quad 2y\frac{dy}{dt} = -2x\frac{dx}{dt}, \quad \frac{dy}{dt} = \frac{-2x}{2y}\frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{x}{y}(-2) = \frac{2x}{y}$$

$x^2+y^2 = 25$. When $x=4$ then $y^2 = 25-16 = 9$, $y = 3$

$$\therefore \quad \left.\frac{dy}{dt}\right|_{x=4} = \frac{(2)(4)}{(3)} = \frac{8}{3}$$

[2] 7. Determine k so that the function

$$g(x) = \begin{cases} 2kx^2 - 1\,, & x \le 1 \\ x^3 + kx\,, & x > 1 \end{cases}$$

is continuous at $x = 1$.

$$2k - 1 = 1 + k$$

$$k = 2$$

[3] 8. Evaluate the following limits.

(a) $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$. 0

(b) $\lim_{t \to \infty} \left(2t \sin\left(\frac{1}{t}\right)\right)$. 2

(c) $\lim_{x \to 0} \left(\frac{\cos(2x) - 1}{\cos(3x) - 1}\right)$. $\frac{4}{9}$

[4] 9.

(a) If $\log_2(x-1) = 4$ then $x =$ 17 .

(b) If $f(x) = |3x|$ then $f'(-4) =$ -3 .

(c) If $f(x) = \dfrac{1}{x+2}$, $g(x) = x+3$, then $(f \circ g)(x) =$ $\frac{1}{x+5}$.

[4] 10. Answer T (true) or F (false) for each of the following questions.

(a) If f is differentiable then f is continuous. T

(b) If $\dfrac{\pi}{4} < x < \dfrac{\pi}{2}$ then $\tan x < 1$. F

(c) A function $f(x)$ cannot be simultaneously even and odd. F

(d) The graph of $y = \dfrac{2x^2 - 5x + 7}{4x^2 - 9x + 21}$ has the line $y = \dfrac{1}{2}$ as a horizontal asymptote.

T

[40]