UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 251 — L04 Fall 2004

MIDTERM EXAM [October 29, 2004 (Fri)]

Time: 50 minutes. PLEASE write your Name on the very last page.
NO CALCULATORS.
Total Marks = 100. Work all problems. Marks are shown in brackets.

Student ID: ____________________

[Marks]

[10] 1. Solve the following inequality for $x \in \mathbb{R}$ and express the solution set in interval notations:

$$\frac{1}{x+1} \leq \frac{3}{x-2}.$$

This inequality is defined for $x \in \mathbb{R} \setminus \{-1, 2\}$

For $x \in \mathbb{R} \setminus \{-1, 2\}$, it is equivalent to

$$\frac{1}{x+1} - \frac{3}{x-2} \leq 0 \Leftrightarrow \frac{x-2-3x-3}{(x+1)(x-2)} \leq 0 \Leftrightarrow \frac{-5-2x}{(x+1)(x-2)} \leq 0$$

$$\Leftrightarrow \frac{5+2x}{(x+1)(x-2)} \geq 0 \Leftrightarrow x \in \left[-\frac{5}{2}, -1\right) \cup (2, +\infty)$$

So $S = \left[-\frac{5}{2}, -1\right) \cup (2, +\infty)$ is the solution set

[12] 2. Find the derivative $y'(x) = \frac{dy}{dx}$ for $y = \sin(4x)\sin^4(x)$.

$$y' = (\sin(4x)\sin^4(x))' = 4\cos(4x)\sin^4(x) + 4\sin^3(x)\cos(x)\sin(4x)$$

$$= 4\sin^3(x)(\cos(4x)\sin(x) + \sin(4x)\cos(x))$$

$$= 4\sin^3(x)\sin(5x)$$

3. Let $f(x) = x^2 + x\cos(x)$.

[15] (a) Find the linear approximation of $f(x)$ at the point $x_0 = \frac{\pi}{2}$.

$f(\frac{\pi}{2}) = \frac{\pi^2}{4}$, $f'(x) = 2x + \cos x - x\sin x$

$\Rightarrow f'(\frac{\pi}{2}) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$.

So the linear approximation of $f(x) = x^2 + x\cos(x)$ at $x_0 = \frac{\pi}{2}$ is $L(x) = f(x_0) + f'(x_0)(x - x_0)$, i.e

$$L(x) = \frac{\pi^2}{4} + \frac{\pi}{2}\left(x - \frac{\pi}{2}\right) = \frac{\pi}{2}x$$

$$\boxed{L(x) = \frac{\pi}{2}x}$$

[6] (b) Find the error in the linear approximation of part (a) at $x = -\frac{\pi}{2}$.

$L(-\frac{\pi}{2}) = -\frac{\pi^2}{4}$, $f(-\frac{\pi}{2}) = \frac{\pi^2}{4}$.

The error is $E(x) = |f(x) - L(x)|$,

at $x = -\frac{\pi}{2}$, $E(-\frac{\pi}{2}) = \left|\frac{\pi^2}{4} + \frac{\pi^2}{4}\right| = \frac{\pi^2}{2}$

[15] 4. Find the equation of the tangent line to the graph of the curve

$$y^3 + y\,x^2 - y^2 + x^2 = 2$$

at the point $(1, 1)$.

$$y = 1 + y'\Big|_{\substack{x=1\\y=1}}(x-1)$$

$$\frac{d}{dx}\left(y^3 + yx^2 - y^2 + x^2\right) = 0$$

$$\left(3y^2 + x^2 - 2y\right)y' + 2yx + 2x = 0$$

At $(1,1)$ we have

$$(3+1-2)\,y'\Big|_{\substack{x=1\\y=1}} + 2 + 2 = 0$$

$$2y' + 4 = 0 \Rightarrow y' = -2$$

So $y = 1 - 2(x-1) = -2x + 3$

$$\boxed{y = -2x + 3}$$

5. Let $f(x)$ be the function defined piecewise by

$$f(x) := \begin{cases} x^2+1 & \text{if} \quad x \le 1 \\ 2x & \text{if} \quad x > 1. \end{cases} \tag{1}$$

[6] (a) What is the value of the limit $\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h}$?

By definition $= f_-'(1) = \frac{d}{dx}(x^2+1)\Big|_{x=1} = 2$

[6] (b) What is the value of the limit $\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h}$?

By definition $= f_+'(1) = \frac{d}{dx}(2x)\Big|_{x=1} = 2$

[8] (c) Is the function f differentiable at $x_0 = 1$? Explain your answer.

Yes. Since the derivatives from left and right exist at $x=1$ and are equal, f is differentiable at $x=1$, and we have

$$f'(1) = f_-'(1) = f_+'(1) = 2$$

[6] (d) Is the function f continuous at $x_0 = 1$? Explain your answer.

Yes. Since f is differentiable at $x_0 = 1$, f is also continuous at $x_0 = 1$. Or $\lim_{x\to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x\to 1} f(x) = 2$

[8] 6. Prove that the equation $x^3 - 4x + 1 = 0$ has at least one solution in the inteval $[1, 2]$.

The function $f(x) := x^3 - 4x + 1$ is a polynomial, so it is everywhere continuous. In particular $f(x)$ is continuous on the interval $[1,2]$.

$f(1) = 1 - 4 + 1 = -2 < 0$

$f(2) = 8 - 8 + 1 = 1 > 0$

So f changes sign on $[1,2]$. By the mean-value theorem we conclude that f admits at least one solution in $[1,2]$

[8] 7. Find the (x, y) coordinates of a point on the graph of the parabola $y = x^2$ for which the tangent line has slope $m = 2$.

Let (x_0, y_0) be the coordinates of that point on the graph of $y = x^2$.

We have $y_0 = x_0^2$.

$y = x^2 \Rightarrow y' = 2x$. The slope of the tangent line at x_0 is $m = 2x_0$. But this is 2. So $2 = 2x_0$. Thus $x_0 = 1$, and $y_0 = 1^2 = 1$.

$(1, 1)$ are the coordinates of that point.

Name:	Student ID:	Marks: