Practice Problems S4

P1. Use the definition of the derivative (i.e., as a limit) to differentiate the following functions:
(a) $f(x) = \frac{1}{\sqrt{2x+1}}$; (b) $f(x) = \frac{1}{x}$.

P2. Given

$$f(x) = \begin{cases} x^3 + 1, & x \geq -1 \\ -3x^2 - 3x, & x < -1 \end{cases}.$$

(a) Find $f(-1)$;

(b) Find $\lim_{h \to 0^-} \frac{f(-1+h)-f(-1)}{h}$ and $\lim_{h \to 0^+} \frac{f(-1+h)-f(-1)}{h}$;

(c) Is $f(x)$ differentiable at $x = -1$? Is $f(x)$ continuous?

P3. Find the equation of the tangent line to the graph of $f(x) = \cos(x)$ at $x = \frac{\pi}{4}$.

P4. Find $f'(x) = \frac{d}{dx} f(x)$ if

(a) $f(x) = \frac{\sin(x)}{\cos(x)+x\sin(x)}$; (b) $f(x) = x^3 \tan^2(x^4)$

(c) $f(x) = \left(1 + \csc(x)\right)^2$; (d) $f(x) = \sqrt[7]{x^4}$.

P5. Find the equation of the tangent line to the curve described by the equation $x^2y - xy^2 = 2\cos(y-1)$ at $(-1, 1)$.

P6. Let $f(x) = \sqrt[3]{x}$.

(a) Find the linear approximation of $f(x)$ at $x = 1$;

(b) Use this approximation to estimate $\sqrt[3]{1.09}$;

(c) Use an appropriate linear approximation of $f(x) = \sqrt[3]{x}$ to estime the value of $\sqrt[3]{0.99}$.

Solutions

P1. (a) $f(x) = \dfrac{1}{\sqrt{2x+1}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2(x+h)+1}} - \frac{1}{\sqrt{2x+1}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sqrt{2x+1}-\sqrt{2(x+h)+1}}{\sqrt{2(x+h)+1}\sqrt{2x+1}}}{h} = \lim_{h\to 0} \frac{\sqrt{2x+1}-\sqrt{2(x+h)+1}}{h\sqrt{2(x+h)+1}\sqrt{2x+1}}$$

$$= \lim_{h\to 0} \frac{2\cancel{x}+\cancel{1}-2(\cancel{x}+h)\cancel{-1}}{h\sqrt{2(x+h)+1}\sqrt{2x+1}\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}\sqrt{2(x+h)+1}\sqrt{2x+1}\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{-2}{\sqrt{2(x+h)+1}\sqrt{2x+1}\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)}$$

$$= \frac{-2}{\sqrt{2x+1}\sqrt{2x+1}\left(\sqrt{2x+1}+\sqrt{2x+1}\right)}$$

$$= \frac{-2}{(2x+1)(2\sqrt{2x+1})} = \frac{-1}{(2x+1)\sqrt{2x+1}}$$

$$\Rightarrow \left(\frac{1}{\sqrt{2x+1}}\right)' = \frac{-1}{(2x+1)\sqrt{2x+1}}$$

(5) $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

$$= \lim_{w \to x} \frac{\frac{1}{w} - \frac{1}{x}}{w - x} = \lim_{w \to x} \frac{\frac{x - w}{wx}}{w - x}$$

$$= \lim_{w \to x} \frac{x - w}{wx(w - x)} = \lim_{w \to x} \frac{-1}{wx}$$

$$= -\frac{1}{x^2}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

P2. $f(x) = \begin{cases} x^3 + 1, & x \geq -1; \\ -3x^2 - 3x, & x < -1 \end{cases}$

(a) $f(-1) = (-1)^3 + 1 = -1 - 1 = 0$

(b) $\lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} \quad \left(= f'_+(-1)\right)$

$$= \lim_{h \to 0^+} \frac{(-1+h)^3 + 1 - 0}{h} = \lim_{h \to 0^+} \frac{\cancel{-1} + 3h - 3h^2 + h^3 \cancel{+1}}{h}$$

$$= \lim_{h \to 0^+} \frac{h^3 - 3h^2 + 3h}{h} = \lim_{h \to 0^+} h^2 - 3h + 3 = 3$$

$$\left(f'_-(x)=\right)\lim_{h\to 0^-}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0^-}\frac{-3(-1+h)^2-3(-1+h)-0}{h}$$

$$=\lim_{h\to 0^-}\frac{-3(\cancel{1}-2h+h^2)-3(\cancel{-1}+h)}{h}$$

$$=\lim_{h\to 0^-}\frac{6h-3h^2-3h}{h}=\lim_{h\to 0^-}3-3h=3$$

(c) Since $f'_-(-1)$ and $f'_+(1)$ exists and are equal to 3, f is differentiable at $x=-1$ with $f'(-1)=f'_-(-1)=f'_+(-1)=3$.

The differentiability of $f(x)$ at $x=-1$ implies that $f(x)$ is also continuous at -1.

P3. The equation of the tangent line at x_0 is given by

$$y-f(x_0)=f'(x_0)(x-x_0)\,,\quad x_0=\frac{\pi}{4}.$$

$$f(x)=\cos x\implies f\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}.$$

$$f'(x)=-\sin x\implies f'(\pi/4)=-\sin\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$$

$$\implies \boxed{y=-\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)+\frac{\sqrt{2}}{2}}$$

P4.

(a) $$\left(\frac{\sin x}{\cos x + x\sin x}\right)' = \frac{\cos x(\cos x + x\sin x) - \sin x(\cancel{-\sin x + \sin x} + x\cos x)}{(\cos x + x\sin x)^2}$$

$$= \frac{\cos^2 x + \cancel{x\sin x\cos x - x\sin x\cos x}}{(\cos x + x\sin x)^2}$$

$$= \frac{\cos^2 x}{(\cos x + x\sin x)^2}$$

(b) $f(x) = x^3\tan^2(x^4)$

$$\Rightarrow f'(x) = (x^3)'\tan^2(x^4) + x^3(\tan^2(x^4))'$$

$$= 3x^2\tan^2(x^4) + 2x^3\tan(x^4)\sec^2(x^4)(x^4)'$$

$$= 3x^2\tan^2(x^4) + 8x^6\tan(x^4)\sec^2(x^4)$$

(c) $$\left[(1+\csc(x))^2\right]' = 2(1+\csc x)(\csc(x))'$$

$$= 2(1+\csc x)(-\cot x\csc(x))$$

(d) $$\left(\sqrt[3]{x^4}\right)' = \left(x^{4/3}\right)' = \tfrac{4}{3}x^{4/3-1} = \tfrac{4}{3}x^{1/3} = \tfrac{4}{3}\sqrt[3]{x^1}$$

P5. $\frac{d}{dx}(x^2y - xy^2) = \frac{d}{dx}(2\cos(y-1))$

$$2xy + x^2y' - y^2 - 2xyy' = -2y'\sin(y-1)$$

at $(-1,1)$, we have

$$2(-1)(1) + (-1)^2 y'\big|_{(-1,1)} - (1)^2 - 2(-1)(1)\, y'\big|_{(-1,1)}$$

$$= -2\, y'\big|_{(-1,1)} \sin(0)$$

$$-2 + y'\big|_{(-1,1)} - 1 + 2y'\big|_{(-1,1)} = 0$$

$$\Rightarrow y'\big|_{(-1,1)} = 3/3 = 1$$

The tangent line to the curve at $(-1,1)$ is $y - y_0 = y'\big|_{(x_0,y_0)}(x - x_0)$ where $(x_0,y_0) = (-1,1)$

$$y - 1 = 1.(x+1), \text{ i.e. } \boxed{y = x + 2}$$

P6. $f(x) = \sqrt[3]{x}$

(a) The linear approximation of $f(x)$ at $x_0 = 1$ is given by $L(x) = f'(x_0)(x_0 - x_0) + f(x_0)$

$$f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f(x_0) = f(1) = 1$$

$$f'(x) = (x^{1/3})' = \frac{1}{3} x^{1/3 - 1} = \frac{1}{3} x^{-2/3} \Rightarrow f'(1) = \frac{1}{3}$$

So, $L(x) = \frac{1}{3}(x-1) + 1$

$$\boxed{L(x) = \tfrac{1}{3} x + \tfrac{2}{3}}$$, i.e.

$L(x) = \frac{1}{3} x + \frac{2}{3} \approx f(x)$ for x close to 1

(b) Since 1.09 is close to 1, we have

$$\sqrt[3]{1.09} = f(1.09) \approx L(1.09) = \frac{1}{3}(1.09 - 1) + 1$$
$$= 1 + \frac{1}{3}(0.09)$$
$$= 1 + 0.03 = 1.03$$

(c) 0.99 is also close to 1. We can use again the linear approximation of f at $x_0 = 1$:

$$\sqrt[3]{0.99} = f(0.99) \approx L(0.99) = \frac{1}{3}(0.99 - 1) + 1$$
$$= 0.99666\ldots$$