Practice Problems S 5

1. Solve the following equations for x:

(a) $3e^{-2x} = 5$; (b) $\ln(4x) - 3\ln x^2 = \ln 2$

2. Find the derivatives of

(a) $y = \ln(2 + \sin(x))$; b) $y = e^{x + e^{-3x}}$

(c) $y = \pi^{\sin x}$; (d) $y = x^{\cos x}$

3. Use L'Hôpital's rule to evaluate:

(a) $\lim_{x \to 0} \frac{x e^x}{1 - e^x}$; (b) $\lim_{x \to 2} \frac{4x - 8}{\sqrt{2x+5} - \sqrt{x^2+5}}$

(d) $\lim_{x \to 0} \left(\csc(x) - \frac{1}{x}\right)$; (c) $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos 3x}{x^2}\right)$

4. Let $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 8$. Find all intervals on which f is increasing, decreasing, concave up or concave down. If any, find all relative extrema and all inflection points.

Solutions

1\. (a) $3e^{-2x} = 5 \Leftrightarrow e^{-2x} = 5/3$

$\Leftrightarrow -2x = \ln(5/3)$

$\Leftrightarrow x = -\frac{1}{2}\ln(5/3)$

(b) $\ln(4x) - 3\ln(x^2) = \ln 2$. This equation is defined for $x > 0$; i.e, any solution must be greater than 0.

$\Leftrightarrow \ln\left(\frac{4x}{x^6}\right) = \ln 2$

$\Leftrightarrow \frac{4}{x^5} = 2 \Leftrightarrow \boxed{x = \sqrt[5]{2}}$

2\. (a) $y = \ln(2+\sin x) \Rightarrow y' = \frac{(2+\sin x)'}{2+\sin x}$

$= \frac{\cos x}{2+\sin x}$

(b) $y = e^{x+e^{-3x}} \Rightarrow y' = (x+e^{-3x})' e^{x+e^{-3x}}$

$= (1-3e^{-3x}) e^{x+e^{-3x}}$

(c) $y = \pi^{\sin x} \Rightarrow y' = (\sin x)' \pi^{\sin x} \ln \pi$

$= \cos x \, \pi^{\sin x} \ln \pi$

(d) $y = x^{\cos x} \Rightarrow \ln y = \cos x \ln x$

$\Rightarrow y'/y = -\sin x \ln x + \frac{\cos x}{x}$

$y' = \left(-\sin x \ln x + \frac{\cos x}{x}\right) x^{\cos x}$

3. (a) $\lim\limits_{x\to 0} \dfrac{xe^x}{1-e^x} = \text{"}\dfrac{0}{0}\text{"}$

$$\overset{\text{L'Hôpital's rule}}{=} \lim_{x\to 0} \frac{(xe^x)'}{(1-e^x)'} = \lim_{x\to 0} \frac{e^x + xe^x}{-e^x} = -1.$$

(b) $\lim\limits_{x\to 2} \dfrac{4x-8}{\sqrt{2x+5}-\sqrt{x^2+5}} = \text{"}\dfrac{0}{0}\text{"}$

$$\overset{\text{L'Hôpital's rule}}{=} \lim_{x\to 2} \frac{(4x-8)'}{(\sqrt{2x+5}-\sqrt{x^2+5})'} = \lim_{x\to 2} \frac{4}{\frac{2}{2}(2x+5)^{-1/2} - x(x^2+5)^{-1/2}}$$

$$= \lim_{x\to 2} \frac{4}{(2x+5)^{-1/2} - x(x^2+5)^{-1/2}} = \frac{4}{9^{-1/2} - 2(9)^{-1/2}}$$

$$= \frac{4}{1/3 - 2/3} = -12$$

(c) $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{\cos 3x}{x^2}\right) = \text{"}\infty - \infty\text{"}$

$$= \lim_{x\to 0} \frac{1-\cos 3x}{x^2} = \text{"}\frac{0}{0}\text{"}$$

$$\overset{\text{L'Hôpital's rule}}{=} \lim_{x\to 0} \frac{(1-\cos 3x)'}{(x^2)'} = \lim_{x\to 0} \frac{3\sin 3x}{2x} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 0} \frac{(3\sin 3x)'}{(2x)'} = \lim_{x\to 0} \frac{9\cos 3x}{2}$$

$$= 9/2$$

$$f''(x) = 2(x-1)$$

x	$-\infty$		1		$+\infty$
$f''(x)=2(x-1)$		$-$	0	$+$	
		$\cap\cap\cap$		$\cup\cup\cup\cup$	

f is concave up on $(1, +\infty)$ and down on $(-\infty, 1)$.

$f'(x)$ changes its sign at -1 from $+$ to $-$
so f has a relative maximum at $x=-1$

$f'(x)$ changes sign again at 3 from $-$ to $+$
so f has a relative minimum at $x=3$.

Or since -1 and 3 are critical points, we can just check the sign for $f''(-1)$ and $f''(3)$.

We have $f''(-1) = 2(-1-1) = -4 < 0$
$\Rightarrow$ relative max at -1

$f''(3) = 2(3-1) = 4 > 0$
$\Rightarrow$ rel. min. at $x=3$

f changes its concavity at 1 from down to up, so f has an inflection point at $x=1$