2006 Fall

Math 251 (L06)

Final Review Solution

1. (i) $\lim_{x\to\infty} \frac{\sqrt{x^2-1}-\sqrt{x^2+x}}{3x} \cdot \frac{\sqrt{x^2-1}+\sqrt{x^2+x}}{\sqrt{x^2-1}+\sqrt{x^2+x}}$

$= \text{"} \quad \frac{(x^2-1)-(x^2+x)}{3x(\sqrt{x^2-1}+\sqrt{x^2+x})}$

$= \text{"} \quad \frac{-1-x}{3x(\sqrt{x^2-1}+\sqrt{x^2+x})} = \lim_{x\to\infty} \frac{-1-x}{3x^2\left(\sqrt{1-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x}}\right)}$

$= 0.$

(ii) $\lim_{x\to 0^-} \frac{1}{x} - \frac{\sqrt{1-2x}}{x}$

$= \text{"} \quad \frac{1-\sqrt{1-2x}}{x} \cdot \frac{1+\sqrt{1-2x}}{1+\sqrt{1-2x}}$

$= \text{"} \quad \frac{1-(1-2x)}{x(1+\sqrt{1-2x})} = \lim_{x\to 0^-} \frac{2\cancel{x}}{\cancel{x}(1+\sqrt{1-2x})} = \frac{2}{2} = 1$

(iii) $\lim_{x\to 0} \frac{x}{\sin x - \ln(1+x)}$

$= \text{"} \quad \frac{1}{\frac{\sin x}{x} - \frac{\ln(1+x)}{x}}$

$= \text{"} \quad \frac{1}{1-\ln(1+x)^{\frac{1}{x}}} = \frac{1}{1-\ln e} = \frac{1}{0} = \infty$

2. A fn is cts at $x=-1$ if $f(-1) = f(-1^+)$.

$\therefore (c-6)(-1) = c^2 \cdot 1$

$\therefore c^2+c-6=0 \quad \therefore (c+3)(c-2)=0$

$\therefore c = -3, 2.$

3.(i) $y = (\sin x)^{\cos x}$ ①

$\ln y = \cos x \ln(\sin x)$

$\therefore \frac{y'}{y} = -\sin x \ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x}$.

$\therefore y' = y\left[-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right]$ ②

$\therefore y'' = y'[\qquad] + y\left[-\cos x \ln(\sin x) - \sin x \frac{\cos x}{\sin x} + \frac{\sin x(2\cos x \cdot(-\sin x)) - \cos^3 x}{\sin^2 x}\right]$,

given in ② given in ①. Simplify.

(ii) $y = e^x \cos 2x$

$\therefore y' = e^x \cos 2x - 2e^x \sin 2x$

$= e^x(\cos 2x - 2\sin 2x)$

$\therefore y'' = e^x\left[\cos 2x - 2\sin 2x - 2\sin 2x - 4\cos 2x\right]$

$= e^x\left[-3\cos 2x - 4\sin 2x\right]$

(iii) $y = x^{\ln 3x}$. ①

$\therefore \ln y = \ln 3x \ln x = (\ln 3 + \ln x)\ln x$

$= (\ln 3)\ln x + (\ln x)^2$

$\therefore \frac{y'}{y} = \frac{\ln 3}{x} + \frac{2\ln x}{x} = \frac{\ln 3 + 2\ln x}{x}$

$$\therefore y' = y\left(\frac{\ln 3 + 2\ln x}{x}\right) \quad (2)$$

$$\therefore y'' = y'\left(\frac{\ln 3+2\ln x}{x}\right) + y\,\frac{x[0+\frac{2}{x}]-[\ln 3+2\ln x]\cdot 1}{x^2}$$

$$= \underbrace{y'}_{\text{given in (2)}}\left(\frac{\ln 3+2\ln x}{x}\right) + \underbrace{y}_{\text{given in (1)}}\,\frac{2-(\ln 3+2\ln x)}{x^2}$$

(iv) $e^y = x^3 \ln(\sin x)$

(1) $\therefore e^y \cdot y' = 3x^2 \ln(\sin x) + x^3 \dfrac{\cos x}{\sin x} = 3x^2 \ln(\sin x) + x^3 \cot x$

(2) $\therefore e^y(y'+y'') = 6x \ln(\sin x) + 3x^2 \cot x - x^3 \operatorname{cosec}^2 x$

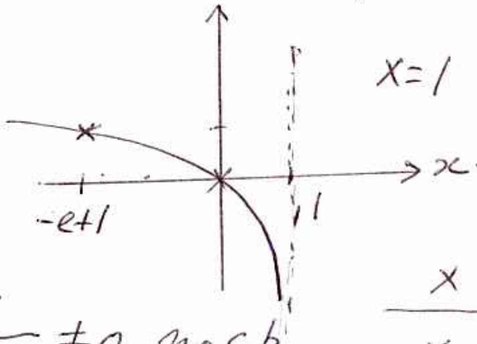
(2) − (1):

$$e^y y'' = (6x-3x^2)\ln(\sin x) + 3x^2 \cot x - x^3 \cot - x^3 \operatorname{cosec}^2 x$$

where e^y is given earlier.

4. Graph:

(i) $y = \ln(1-x)$



$x=1$ is Vertical asymptote

Domain: $1-x>0$

$\therefore x<1$

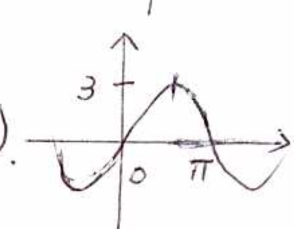
$y' = \dfrac{-1}{1-x} = \dfrac{1}{x-1} \neq 0$ no c.p.

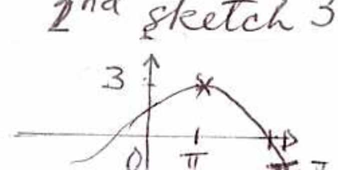
x	0	$1-e$
y	$\ln 1 = 0$	$\ln e = 1$

$y'' = \dfrac{-1}{(x-1)^2} < 0$ (concave down)

(ii) $y = 3\sin(x - \frac{\pi}{3})$

1st, Sketch $3\sin x$, 2nd sketch $3\sin(x-\frac{\pi}{3})$.





(iii) $y = \frac{3x}{1-x^2}$ Domain: $1-x^2 \neq 0$

$x^2 \neq 1 \therefore x \neq \pm 1$.

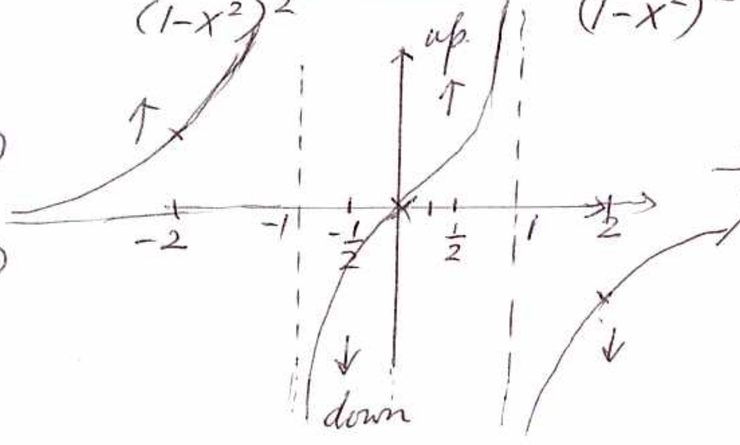
$y(0) = 0$

$$y' = \frac{(1-x^2)3 - 3x(-2x)}{(1-x^2)^2} = \frac{3(1+x^2)}{(1-x^2)^2} > 0 \text{ (no c.p.)}$$

Horiz. Asymp.

$\lim_{x\to\infty} y = 0$

$\lim_{x\to-\infty} y = 0$



x	$\frac{1}{3}$	$\frac{1}{2}$	2	-2
y	$\frac{9}{8}$	2	-2	2

5. (i). $y = \frac{x^2-1}{x^2-4}$ Domain: $x^2-4 \neq 0$

$x \neq \pm 2$.

$$y' = \frac{(x^2-4)2x - (x^2-1).2x}{(x^2-4)^2}$$

$$= \frac{-6x}{(x^2-4)^2}$$

Horiz. asymp. $y = \lim_{x\to\pm\infty} \frac{x^2-1}{x^2-4} = 1$.

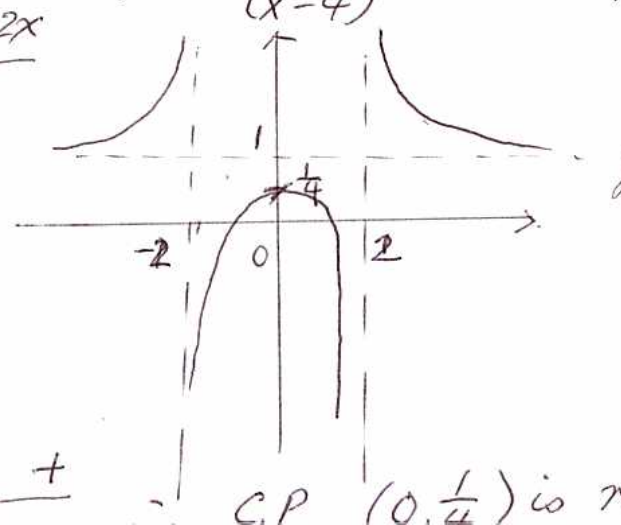
Critical pts: $y' = \frac{-6x}{(x^2-4)^2} = 0 \therefore x = 0$ C.P.

$+ \; y = \frac{1}{4}$.

$$y'' = \frac{(x^2-4)^2(-6) + 6x \cdot 2(x^2-4)2x}{(x^2-4)^4}$$

$$= \frac{(x^2-4)[-6(x^2-4) + 24x^2]}{(x^2-4)^4}$$

$$= \frac{6(3x^2+4)}{(x^2-4)^3}$$



y'' + − +

C.P. $(0, \frac{1}{4})$ is rel. max. pt.

5 (ii) $y = \dfrac{x}{1-4x^2}$ Domain $4x^2 \neq 1$

$x^2 \neq \frac{1}{4}$

$x \neq \pm\frac{1}{2}$.

$\therefore$ Vertical asymp. $x = \frac{-1}{2}, x = \frac{1}{2}$

Horiz. asymp. $y = \lim\limits_{x\to\infty} \dfrac{x}{1-4x^2} = 0$.

$$y' = \frac{(1-4x^2)\cdot 1 - x(-8x)}{(1-4x^2)^2} = \frac{1+4x^2}{(1-4x^2)^2} > 0 \quad \therefore \text{no c.p.}$$

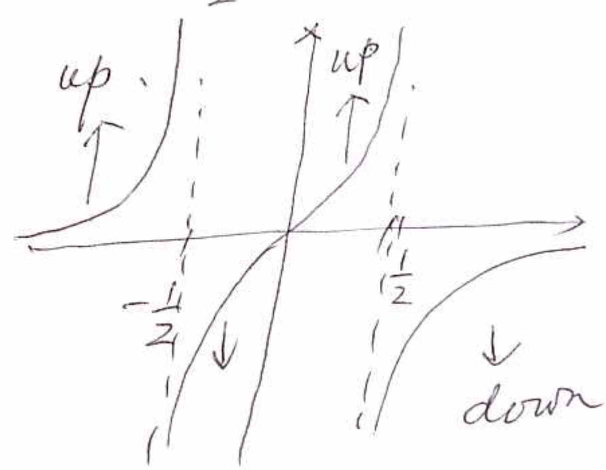
$$y'' = \frac{(1-4x^2)^2 8x - (1+4x^2)\cdot 2(1-4x^2)(-8x)}{(1-4x^2)^4}$$

$$= \frac{8x(1-4x^2)\left[(1-4x^2) + 2(1+4x^2)\right]}{(1-4x^2)^4}$$

$$= \frac{8x(3+4x^2)}{(1-4x^2)^3}$$

y''	+		−		+		−
		$-\frac{1}{2}$		0		$\frac{1}{2}$	

Similar to prob. 4 (iii).



5(iii) $y = 1 - \dfrac{1}{(1-x)(2+x)}$ Domain: $x \neq 1$, $x \neq -2$

$\therefore$ Vert. asymp $x=1$, $x=-2$.

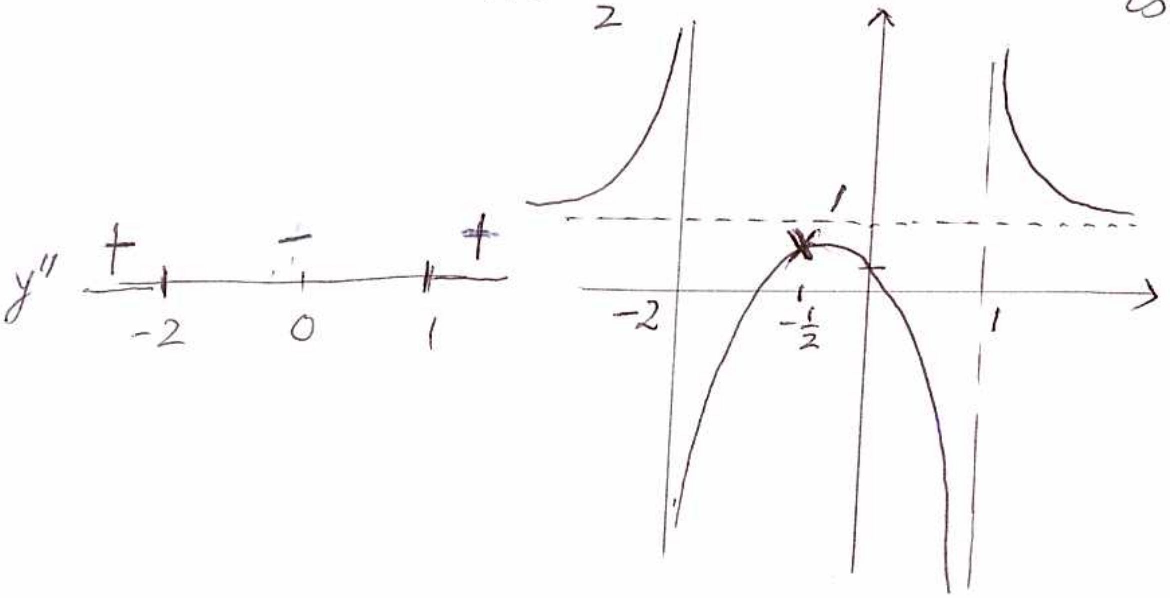
Horiz. asymp. $y = \lim_{x\to\infty} 1 - \dfrac{1}{(1-x)(2+x)} = 1$.

$y' = \dfrac{-1-2x}{[(1-x)(2+x)]^2}$ (after simplifying)

$= 0$ for c.p. $\therefore$ $2x = -1$, $x = -\frac{1}{2}$ & $y = \frac{5}{9}$. c.p.

$y'' = \dfrac{-2(2+3x+3x^2)}{[(1-x)(2+x)]^3}$ (after working out)

$\therefore$ $y''\Big|_{x=-\frac{1}{2}} = -ve$ $\therefore$ c.p. $x=-\frac{1}{2}$, $y=\frac{5}{9}$ is rel. max.



6. (i) $\int 4 - \frac{1}{x^3} + \frac{1}{4x} - \cos 3x + \tan 2x \, dx$

$= 4x + \frac{1}{2x^2} + \frac{1}{4}\ln|x| - \frac{1}{3}\sin 3x + \frac{1}{2}\ln \sec 2x + c$

(Check by differentiation)

(ii) $\int (\sin 2x + \cos 2x)^2 \, dx$

$= \int \sin^2 2x + 2\sin 2x \cos 2x + \cos^2 2x \, dx$

$= \int 1 + 2\sin 2x \cos 2x \, dx$

$= x + \frac{1}{2}\sin^2 2x + C.$

Use $\int f f' \, dx = \frac{1}{2} f^2$

(iii) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx = \ln(e^x + e^{-x}) + C$

$\int \frac{f'}{f} dx = \ln f.$

7 (i). $\int (e^{2x} + e^{-2x})^2 dx$

$= \int e^{4x} + 2\underbrace{e^{-2x} e^{-2x}}_{=1} + e^{-4x} \, dx.$

$= \frac{1}{2}e^{4x} + 2x - \frac{1}{4}e^{-4x} + C.$

(ii) $\int e^{\sin x} \cos x \, dx$

Either use:

$\int e^f f' \, dx = e^f$

Let $u = \sin x$

$\therefore du = \cos x \, dx.$

$\therefore \int e^{\sin x} \cos x \, dx = \int e^u . du = e^u + C$

$= e^{\sin x} + C.$

7 (iii) $I = \int \cos 3t \sin^2 3t \, dt$

Either use: $\int f' f^2 \, dt = \frac{f^3}{3}$

Let $u = \sin 3t$

$\therefore du = 3\cos 3t \, dt$

ie $\frac{du}{3} = \cos 3t \, dt$

$\therefore I = \int u^2 \frac{du}{3} = \frac{1}{3}\frac{u^3}{3} + C = \frac{1}{9}\sin^3 3t + C.$

(iv). $I = \int t(1+t)^{1/3} dt$

Let $u = 1+t$ & $u-1 = t$.

$\therefore du = dt$

$\therefore I = \int (u-1) u^{1/3} du$

$= \int u^{4/3} - u^{1/3} \, du$

$= \frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3} + C$

$= \frac{3}{7}(1+t)^{7/3} - \frac{3}{4}(1+t)^{4/3} + C.$

8 (i) Fl. Theorem of Calculus.

$$y = \int_1^{x^2} (1-x^2)^{\frac{1}{3}}\,dx$$

$$\frac{d}{dx} y = \frac{d}{dx}\int_1^{x^2} (1-x^2)^{\frac{1}{3}}\,dx$$

$$= \left[1-(x^2)^2\right]^{\frac{1}{3}} \cdot 2x - 0$$

$$= 2x\left(1-x^4\right)^{\frac{1}{3}}$$

(ii). $$\frac{d}{dx}\int_x^{x^2} (1-\sin t)^{\frac{1}{2}}\,dt$$

$$= \left[1-\sin(x^2)\right]^{1/2} 2x - (1-\sin x)^{\frac{1}{2}} \cdot 1$$

$$= 2x\left(1-\sin(x^2)\right)^{1/2} - (1-\sin x)^{\frac{1}{2}}.$$

(iii). $$\frac{d}{dx}\int_{-x}^{\tan x} (1-\cos 2u)^4\,du$$

$$= \left[1-\cos(2\tan x)\right]^4 \cdot \sec^2 x - (1-\cos 2(-x))^4 \cdot -1$$

$$= \sec^2 x\left(1-\cos(2\tan x)\right)^4 + (1-\cos 2x)^4,$$

since $\cos -2x = \cos 2x$;