Fall 2006

MATH 251 (L06)

Mid-Term Review Solutions

1. (a) $\frac{5x+24}{x} - x \geq 0 \quad \therefore \quad \frac{5x+24-x^2}{x} = \frac{(8-x)(3+x)}{x} \geq 0$

Solutions : $(-\infty, -3] \cup (0, 8]$

(b) If $x<0$, no solution

If $x>0$, we square both sides : $\frac{24^2}{(x-10)^2} < x^2$,

or $24^2 < x^2(x-10)^2$

$24^2 - x^2(x-10)^2 < 0 \qquad a^2 - b^2$

$(24 + x(x-10))(24 - x(x-10)) < 0$

$(x^2 - 10x + 24)(24 + 10x - x^2) < 0$

$(x-6)(x-4)(12-x)(2+x) < 0$

Solutions $\quad (4,6) \cup (12, \infty)$.

2. (a) $\lim\limits_{x \to 0^-} \frac{1}{x} + \sqrt{\frac{1-12x}{x^2}}$

$= \lim\limits_{x \to 0^-} \frac{1}{x} + \frac{\sqrt{1-12x}}{\sqrt{x^2}} \qquad \left(\sqrt{x^2} = |x| = -x, \text{ since } x \to 0^-\right)$

$= \lim\limits_{x \to 0^-} \frac{1}{x} + \frac{\sqrt{1-12x}}{-x}$

$= \lim\limits_{x \to 0^-} \frac{1-\sqrt{1-12x}}{x}$

$= \lim\limits_{x \to 0^-} \frac{\left(1-\sqrt{1-12x}\right)\left[1+\sqrt{1-12x}\right]}{x \left[1+\sqrt{1-12x}\right]}$

$= \lim\limits_{x \to 0^-} \frac{1-(1-12x)}{x\left[1+\sqrt{1-12x}\right]}$

$= \lim\limits_{x \to 0^-} \frac{+12\cancel{x}}{\cancel{x}\left[1+\sqrt{1-12x}\right]} = \frac{12}{1+\sqrt{1}} = 6.$

(b) Use $A-B = (A^{1/3} - B^{1/3})(A^{2/3} + A^{1/3}B^{1/3} + B^{2/3})$.

2(c) $\lim\limits_{x\to-\infty} \dfrac{3x-\sqrt{9x^2-2x}}{x+1}$

$= \lim\limits_{x\to-\infty} \dfrac{3x-\sqrt{9x^2-2x}\;/x}{x+1\;/x}$

$= \text{"} \quad \dfrac{3-\dfrac{\sqrt{9x^2-2x}}{-\sqrt{x^2}}}{1+\dfrac{1}{x}}$

$= \text{"} \quad \dfrac{3+\sqrt{9-\dfrac{2}{x}}}{1+\dfrac{1}{x}}$

$= \quad \dfrac{3+\sqrt{9}}{1} = \sqrt{6}.$

3.(a) A fn $f(x)$ is cts at $x=a$ if $\lim\limits_{x\to a} f(x) = f(a)$.

$\lim\limits_{x\to-4^-} f(x) = \lim\limits_{x\to-4^-} \dfrac{|x+4|}{x+4} = \lim\limits_{x\to-4^-} \dfrac{-(x+4)}{x+4} = -1$ and

$\lim\limits_{x\to-4^+} f(x) = \lim\limits_{x\to-4^+} (4+a)x = (4+a)(-4)$,

$\therefore \lim\limits_{x\to-4} f(x)$ exists if $-1 = (4+a)(-4)$.

$\therefore a = \frac{1}{4} - 4 = \frac{-15}{4}$.

If $a = \frac{-15}{4}$, $\lim\limits_{x\to-4} f(x) = -1 = f(-4)$.

& $\therefore f(x)$ is cts at $x=-4$.

$f(x)$ is a polynomial & is cts everywhere if $a = -\frac{15}{16}$

3(b). Range: $y \geq 0$

Domain:

$1-3x+6x^2 \geq 0$

$1+6(x^2-\frac{x}{2}) \geq 0$

$1+6[(x-\frac{1}{4})^2-\frac{1}{16}] \geq 0$

$1+6(x-\frac{1}{4})^2-\frac{6}{16} \geq 0$

" true for all x.

4. $y = \dfrac{x}{(1-x)(2+x)} = \dfrac{x}{2-x-x^2}$

(i) Domain $x \neq 1$, $x \neq -2$. Vert. asymp. $x=1$, $x=-2$, Horiz. asy. $y=0$

(ii). $y' = \dfrac{2-x-x^2 - x(-1-2x)}{[(1-x)(2+x)]^2} = \dfrac{2+x^2}{(\quad)^2}$

$\therefore y'(2) = \dfrac{6}{4^2} = \dfrac{3}{8}$.

$y'' = \dfrac{(1-x)^2(2+x)^2 \cdot 2x - (2+x^2)[2(2-x-x^2)\cdot(-1-2x)]}{(\quad)^4}$ etc.

$\therefore y''(2) = \dfrac{4^2 \cdot 4 - \cancel{6}[2(-4)(-5)]}{4^4} = \cancel{\dfrac{4^2(4-30)}{4^4}} = \cancel{\dfrac{26}{4^2}} = \dfrac{-11}{16}$.

(iii) Eq of tangent line at $x=2$ is

$y - y_0 = \frac{3}{8}(x-2)$, where $y_0 = \dfrac{2}{-4} = -\dfrac{1}{2}$. etc.

(iv) $y' = \dfrac{2+x^2}{(\quad)^2} > 0$ $\therefore$ There are no pts where $y'=0$.

5. By definition,

$y' = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

(a) $y' = \lim_{h\to 0} \dfrac{1}{h}\left[\dfrac{3(x+h)}{1-(x+h)^2} - \dfrac{3x}{1-x^2}\right]$

$= \text{''} \; \dfrac{1}{h} \; \dfrac{3(\cancel{x}+h)(1-x^2) - 3x[\cancel{1} - (\cancel{x^2}+2xh+h^2)]}{[1-(x+h)^2]\cdot(1-x^2)}$

$= \text{''} \; \dfrac{1}{h} \; \dfrac{3h(1-x^2)+3x(2xh+h^2)}{[\quad]\cdot(\quad)} = \lim_{h\to 0} \dfrac{3(1-x^2)+3x(2x+h)}{[\quad]\cdot(\quad)}$

$= \dfrac{3+3x^2}{(1-x^2)^2}$

(b) Similar.

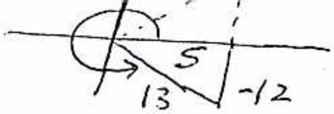
6. (a) $\lim_{x\to 0} \dfrac{\tan 3x}{\sin \frac{x}{2}} = \lim_{x\to 0} \dfrac{3x \sin 3x / 3x}{\frac{x}{2}\cdot \cos 3x \cdot \dfrac{\sin \frac{x}{2}}{\frac{x}{2}}}$

$= \lim_{x\to 0} \dfrac{3x}{\frac{x}{2}} \cdot \dfrac{1}{1\cdot 1} = 6$.

(b) $\lim_{t\to 0} \dfrac{2t + \tan t / t}{3\sin t / t}$

$= \lim_{t\to 0} \dfrac{2 + \dfrac{\sin t}{t \cos t}}{3 \sin t / t} = \dfrac{2 + 1\cdot 1}{3\cdot 1} = 1$.

$\cos x = \frac{5}{13}$



(a) $\sin 2x = 2\sin x \cos x$
$= 2 \cdot \frac{-12}{13} \cdot \frac{5}{13} = \frac{-120}{169}$

(b) $\cos 2x = 2\cos^2 x - 1$
$= 2\left(\frac{5}{13}\right)^2 - 1$
$= \frac{50}{169} - 1 = \frac{-119}{169}$.

(c) $\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{\frac{-120}{169}}{\frac{-119}{169}} = \frac{120}{119}$.

Chain Rule

8. (a) $y = \left(1 + \frac{3}{x} + 2x\right)^{1/2} \sin^2\left(1 + \frac{2}{x}\right)$

$\therefore y' = \frac{1}{2}\left(1+\frac{3}{x}+2x\right)^{-\frac{1}{2}}\left(\frac{-3}{x^2}+2\right)\sin^2\left(1+\frac{2}{x}\right) + \left(1+\frac{3}{x}+2x\right)^{1/2} \cdot 2\sin\left(1+\frac{2}{x}\right) \cdot \cos\left(1+\frac{2}{x}\right) \cdot \left(\frac{-2}{x^2}\right)$

(b) $y' = 2\cos x(-\sin x)\sin^3\left(x^2 - \frac{\pi}{4}x\right) + \cos^2 x \cdot 3\sin^2\left(x^2 - \frac{\pi x}{4}\right) \cos\left(x^2 - \frac{\pi x}{4}\right)\left(2x - \frac{\pi}{4}\right)$

(c) $y' = \frac{6}{\cos(x^2+3x)} - \frac{6x(-\sin(x^2+3x))(2x+3)}{\cos^2(x^2+3x)}$, treating y as a product.

Implicit fn.)

(d). $\frac{d}{dx}$: $\cos(x^2y)\left[2xy + x^2y'\right] + y'x + y = 2yy'$

$\therefore y'\left(x^2\cos(x^2y) + x - 2y\right) = -2xy\cos(x^2y) - y$.

$\therefore y' = \frac{-\cdots}{\cdots}$

(e) $\frac{d}{dx}$: $\sec^2(y+y^2) \cdot (y' + 2yy') = y' - x$.

$\therefore y'\left(\sec^2(y+y^2) \cdot (1+2y) - 1\right) = \cancel{x} - 1$

$\therefore y' = \frac{\cancel{x} - 1}{(1+2y)\sec^2(y+y^2) - 1}$

9. (a) $x^2 + xy^2 = 10$, $x' = 2$

$\frac{d}{dt}$: $2xx' + x'y^2 + x \cdot 2yy' = 0$

(i) For $x = 3$, we get $9 + 3y^2 = 10$ $\therefore y^2 = \frac{1}{3}$ $\therefore y = \pm\frac{1}{\sqrt{3}}$.

$\;\;\;\;$ $6(-2) + 2\left(\frac{1}{3}\right) \pm 2\sqrt{3}\, y' = 0$ $\therefore y' = \frac{12 - \frac{2}{3}}{\pm 2\sqrt{3}}$ etc.

(ii) For $y = 4$, we get $x^2 + 16x - 10 = 0$

$(x+8)^2 - 64 - 10 = (x+8)^2 - 74 = (x+8+\sqrt{74})(x+8-\sqrt{74}) = 0$ $\therefore x = \cancel{2, 8}\ -8-\sqrt{74}, -8+\sqrt{74}$

For $y = 4$, $x = -8-\sqrt{74}$ we get

$y = 4$, $x = -8+\sqrt{74}$ "

(b) $\frac{d}{dt}$: Similar etc.