Name: ______________________________

UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 251 — L06 FALL 2006

November 8, 2006 **MID-TERM TEST** Time: 50 minutes

Solution

You may attempt all problems. Only the best 4 will count. No calculators.

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1. (i) Find the value of a which makes the function

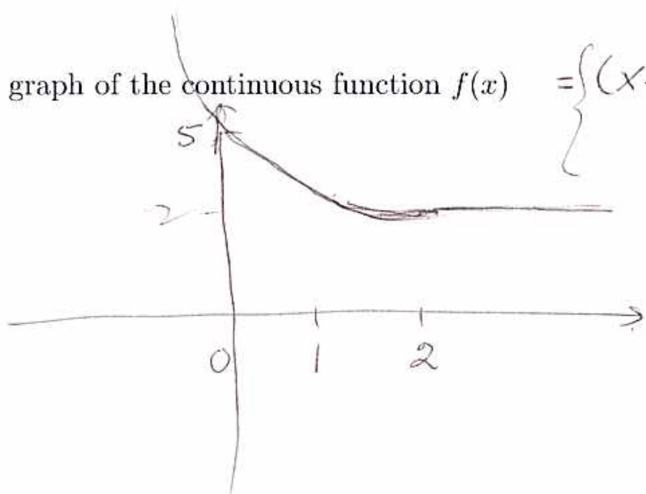
$$f(x) = \begin{cases} (x-2)^2 + a, & \text{if } x \le 2 \\ \dfrac{x-2}{|x-2|}, & \text{if } x > 2 \end{cases}$$

continuous at $x = 2$.

$f(x)$ is cts at $x=2$ if $\lim_{x\to 2} f(x) = f(2) = a$.

$\lim_{x\to 2^+} \frac{x-2}{|x-2|} = \lim_{x\to 2^+} \frac{x-2}{x-2} = \underline{1 = a}$.

(ii) Sketch the graph of the continuous function $f(x)$ $= \begin{cases} (x-2)^2+1, & \text{if } x \le 2 \\ 1, & \text{if } x > 2 \end{cases}$



(i) Use the definition of derivative to find the derivative of the function

$$f(x) = \frac{x}{1-4x}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h}{1-4(x+h)} - \frac{x}{1-4x}\right]$$

$$= \text{〃} \quad \frac{1}{h} \frac{(x+h)(1-4x) - x[1-4(x+h)]}{[1-4(x+h)][1-4x]}$$

$$= \text{〃} \quad \frac{1}{h} \frac{x(1-4x) + h - 4hx - x + 4x^2 + 4xh}{[\quad][\quad]}$$

$$= \frac{1}{(1-4x)^2}.$$

(ii) Find the equation of the tangent line to the above function at $x = \frac{1}{2}$

$$m = \left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = \frac{1}{(1-2)^2} = 1$$

Eq of tgt line is

$$y - \frac{1}{2} = 1\left(x - \frac{1}{2}\right)$$

$$y = x.$$

2. (i) Evaluate $\lim_{x\to-\infty} 2x+\sqrt{4x^2-3x}$

$$= \text{''} \left(2x+\sqrt{4x^2-3x}\right)\frac{\left(2x-\sqrt{\ }\right)}{\left(2x-\sqrt{\ }\right)}$$

$$= \text{''} \frac{4x^2-(4x^2-3x)}{(\quad)}$$

$$= \text{''} \frac{3x}{2x-\sqrt{4x^2-3x}} \Big/ x \text{ or } -\sqrt{x^2}$$

$$= \text{''} \frac{3}{2+\sqrt{4-\frac{3}{x}}} = \frac{3}{2+2} = \frac{3}{4}$$

(ii) Evaluate $\lim_{t\to 0} \frac{2t\sin^2(t)}{\sin^3(3t)}$

$$= \text{''} \frac{2t\,\frac{\sin t}{t}\frac{\sin t}{t}\cdot t^2}{\frac{\sin 3t}{3t}\frac{\sin 3t}{3t}\frac{\sin 3t}{3t}\cdot(3t)^3}$$

$$= \frac{2}{27}\,\frac{1\cdot 1}{1\cdot 1\cdot 1}\,\frac{t^3}{t^3}$$

$$= \frac{2}{27}$$

3. (i) Find y' for $y = \dfrac{\cos(2x + \frac{\pi}{x})}{\sin\left(\frac{\pi}{2} - x^2\right)}$ 2

$\frac{\pi}{2}$

$$y' = \frac{\sin\left(\frac{\pi}{2}-x^2\right)\left(-2\sin\left(2x+\frac{\pi}{2}\right)\right) - \cos\left(2x+\frac{\pi}{2}\right)\cdot 2x\cos\left(\frac{\pi}{2}-\right.}{\sin^2\left(\frac{\pi}{2}-x^2\right)}$$

$$= \frac{-2\sin\left(2x+\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}-x^2\right)} + \frac{2x\cos\left(2x+\frac{\pi}{2}\right)\cot\left(\frac{\pi}{2}-x^2\right)}{\sin\left(\frac{\pi}{2}-x^2\right)}$$

(ii) Find y' for $y = x^2 + 1 - \sec(xy)$

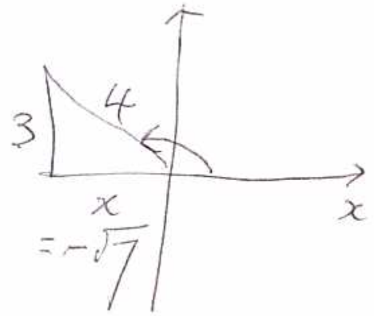
$\frac{d}{dx}$: $$\frac{dy}{dx} = 2x - \sec(xy)\tan(xy)\left[y + x\frac{dy}{dx}\right]$$

$$\therefore \frac{dy}{dx}\left(1 + x\sec(xy)\tan(xy)\right) = 2x - y\sec(xy)\tan(xy)$$

$$\therefore \frac{dy}{dx} = \frac{\quad}{\quad}$$

4. (i) Given $\sin x = \frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, evaluate

(a) $\sin 2x$ (b) $\cos 2x$



$x^2 = 16 - 9 = 7$

$\therefore x = -\sqrt{7}$

(a) $\sin 2x = 2 \sin x \cos x$

$= 2 \cdot \frac{3}{4} \cdot \frac{-\sqrt{7}}{4} = \frac{-3\sqrt{7}}{8}$

(b) $\cos 2x = 1 - 2\sin^2 x$ (or $2\cos^2 x - 1$ or $\cos^2 x - \sin^2 x$)

$= 1 - 2 \cdot \frac{9}{16}$

$= -\frac{1}{8}$

(ii) Given $\sin^2\left(\frac{\pi x}{4}\right) + xy^2 = \frac{3}{2}$ and $\frac{dx}{dt} = 2$, evaluate $\frac{dy}{dt}$ at $x = 1$, $y = 1$.

$\frac{d}{dt}$: $2 \sin\left(\frac{\pi x}{4}\right) \cos\left(\frac{\pi x}{4}\right) \cdot \frac{\pi}{4} \frac{dx}{dt} + \frac{dx}{dt} y^2 + 2xy\frac{dy}{dt} = 0$

(or $\frac{d}{dx}$)

At $x=1$, $y=1$, $\frac{dx}{dt} = 2$

$2 \sin\frac{\pi}{4} \cos\frac{\pi}{4} \cdot \frac{\pi}{4} \cdot 2 + 2 \cdot 1 + 2\frac{dy}{dt} = 0$

$2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\pi}{4} \cdot 2 + 2 + 2\frac{dy}{dt} = 0$

$\frac{\pi}{2} + 2 + 2\frac{dy}{dt} = 0$

$\therefore \frac{dy}{dt} = -\frac{\pi}{4} - 1.$

Student ID Number: ____________ **Grade:** ____________