Sketching Curves

We have developed enough techniques to be able to sketch curves and graphs of functions much more efficiently than before. We shall investigate systematically the behavior of a curve, and the mean value theorem will play a fundamental role.

We shall especially look for the following aspects of the curve:

1. Intersections with the coordinate axes.
2. Critical points.
3. Regions of increase.
4. Regions of decrease.
5. Maxima and minima (including the local ones).
6. Behavior as x becomes very large positive and very large negative.
7. Values of x near which y becomes very large positive or very large negative.

These seven pieces of information will be quite sufficient to give us a fairly accurate idea of what the graph looks like. We shall devote a section to considering one other aspect, namely:

8. Regions where the curve is convex upwards or downwards. This tells us how the curve is bending.

We shall also introduce a new way of describing points of the plane and functions, namely polar coordinates. These are especially useful in connection with the trigonometric functions.

Sketch $y = \frac{x-1}{x+1}$

1. When $x = 0$, we have $f(x) = -1$. When $x = 1, f(x) = 0$.
2. The derivative is

$$f'(x) = \frac{2}{(x+1)^2}.$$

(You can compute it using the quotient rule.) It is never 0, and therefore the function has no critical points.

3. The denominator is a square and hence is always positive. Thus $f'(x) > 0$ for all x. The function is increasing for all x. Of course, the function is not defined for $x = -1$ and neither is the derivative. Thus it would be more accurate to say that the function is increasing in the region

$$x < -1$$

and is increasing in the region $x > -1$.

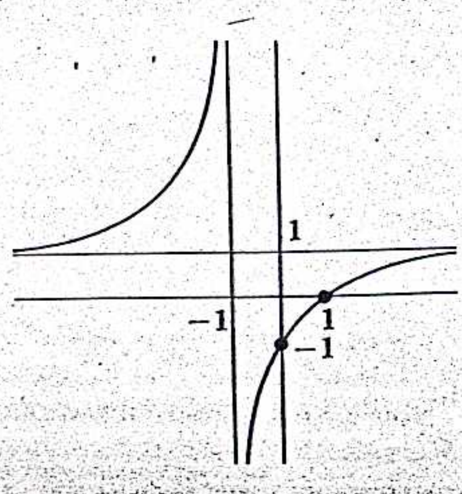
4. There is no region of decrease.
5. Since the derivative is never 0, there is no relative maximum or minimum.
6. As x becomes very large positive, our function approaches 1 (using the method of the preceding section). As x becomes very large negative, our function also approaches 1.

Finally, there is one more useful piece of information which we can look into, when $f(x)$ itself becomes very large positive or negative:

7. As x approaches -1, the denominator approaches 0 and the numerator approaches -2. If x approaches -1 from the right, then the denominator is positive, and the numerator is negative. Hence the fraction

$$\frac{x-1}{x+1}$$

is negative, and is very large negative.



is small when x is close to -1. Putting all this information together, we see that the graph looks like that in the preceding figure.

We have drawn the two lines $x = -1$ and $y = 1$, as these play an important role when x approaches -1 and when x becomes very large, positive or negative.

Example 2. Sketch the graph of the curve

$$y = -x^3 + 3x - 5.$$

1. When $x = 0$, we have $y = -5$.
2. The derivative is

$$f'(x) = -3x^2 + 3.$$

It is 0 when $3x^2 = 3$, which is equivalent to saying that

$$x^2 = 1, \quad \text{or} \quad x = \pm 1.$$

These are the critical points.

3. The derivative is positive when $-3x^2 + 3 > 0$, which amounts to saying that

$$3x^2 < 3 \quad \text{or} \quad x^2 < 1.$$

This is equivalent to the condition

$$-1 < x < 1,$$

which is therefore a region of increase.

4. When $-3x^2 + 3 < 0$, the function decreases. This is the region given by the inequality

$$3x^2 > 3$$

or $x^2 > 1$. Thus when

$$x > 1 \quad \text{or} \quad x < -1,$$

the function decreases.

5. Since the function decreases when $x < -1$ and increases when $x > -1$ (and is close to -1), we conclude that the point -1 is a local minimum. Also, $f(-1) = -7$.

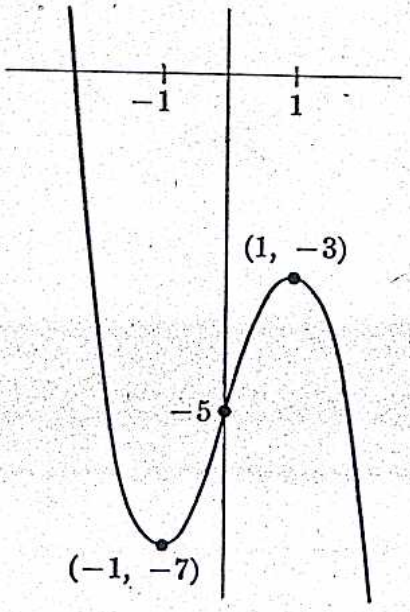
Similarly, the point 1 is a relative maximum and $f(1) = -3$.

6. As x becomes very large positive, x^3 is very large positive and $-x^3$ is very large negative. Hence our function becomes very large negative, as we see if we put it in the form

$$f(x) = -x^3\left(1 - \frac{3}{x^2} + \frac{5}{x^3}\right).$$

Similarly, as x becomes very large negative, our function becomes very

Putting all this information together, we see that the graph looks like this:



EXERCISES

Sketch the following curves, indicating all the information stated in the introduction:

1. $y = \dfrac{x^2 + 2}{x - 3}$
2. $y = \dfrac{x - 3}{x^2 + 1}$
3. $y = \dfrac{x + 1}{x^2 + 1}$
4. $y = \sin^2 x$
5. $y = \cos^2 x$
6. $y = \dfrac{x^2 - 1}{x}$
7. $y = \tan^2 x$
8. $y = \dfrac{x^3 + 1}{x + 1}$
9. $y = x^4 - 2x^3 + 1$
10. $y = \dfrac{2x^2 - 1}{x^2 - 2}$
11. $y = \dfrac{2x - 3}{3x + 1}$
12. $y = x^4 + 4x$
13. $y = x^5 + x$
14. $y = x^6 + 6x$
15. $y = x^7 + x$
16. $y = x^8 + x$
17. Which of the following polynomials have a minimum (for all x)?

(a) $x^6 - x + 2$ (b) $x^5 - x + 2$
(c) $-x^6 - x + 2$ (d) $-x^5 - x + 2$
(e) $x^6 + x + 2$ (f) $x^5 + x + 2$

Sketch the graphs of these polynomials.