Practice Problems S3

P1. Evaluate the following limits if they exit or are infinity (if not explain why).

1. $\lim\limits_{x \to +\infty} \sqrt{4x^2 + x} - \sqrt{4x^2 - x}$;
2. $\lim\limits_{x \to -\infty} (-2x^3 + 3x^2 + x + 10)$;
3. $\lim\limits_{x \to -2} (3x^2 - 2x + 7)$;
4. $\lim\limits_{x \to \frac{3}{2}} \frac{2x-3}{|2x-3|}$;
5. $\lim\limits_{x \to -0} \frac{1}{x\sqrt{x+1}} - \frac{1}{x}$;
6. $\lim\limits_{x \to 2} \frac{4x-8}{\sqrt{2x+5} - \sqrt{x^2+5}}$;
7. $\lim\limits_{x \to +\infty} \frac{-x^3+2x^2+7x+1}{12+3x-4x^3}$;
8. $\lim\limits_{x \to -2^-} \frac{x^2+2x}{x^2-4}$;
9. $\lim\limits_{x \to +\infty} \frac{|x|}{x}$;
10. See recommended problems.

P2. Find the limits (if they exist) at the breakpoints of

$$f(x) = \begin{cases} \frac{1}{x+2}, & x < -2 \\ x^2 - 5, & -2 < x \leq 3 \\ \sqrt{x + 13}, & x > 3. \end{cases}$$

P3. For which values of k is the following function continuous at x=-3:

$$f(x) = \begin{cases} 8 - x^2, & x \geq -3 \\ \frac{k}{x}, & x < -3 \end{cases}.$$

P4. Find an interval of length

(a) 1,

(b) $\frac{1}{2}$

which contains a root of the equation $x^3 + 4x - 7 = 0$.

P5. Prove that the equation $x^3 - 15x + 1 = 0$ has at least two roots in the interval $[0, 4]$.

Solutions

P1. 1. $\lim_{x\to+\infty}\left(\sqrt{4x^2+x}-\sqrt{4x^2-x}\right)$ is an indeterminate limit of the form $+\infty-\infty$. It involves radicals ($\sqrt{A}-\sqrt{B}=\dfrac{A-B}{\sqrt{A}+\sqrt{B}}$). So,

$$\lim_{x\to+\infty}\left(\sqrt{4x^2+x}-\sqrt{4x^2-x}\right)=\lim_{x\to+\infty}\frac{4x^2+x-(4x^2-x)}{\sqrt{4x^2+x}+\sqrt{4x^2-x}}$$

$$=\lim_{x\to+\infty}\frac{2x}{\sqrt{4x^2+x}+\sqrt{4x^2-x}}=\lim_{x\to+\infty}\frac{2x}{\sqrt{4x^2}\left(\sqrt{1+\frac{1}{4x}}+\sqrt{1-\frac{1}{4x}}\right)}$$

$$=\lim_{x\to+\infty}\frac{2x}{2|x|\left(\sqrt{1+\frac{1}{4x}}+\sqrt{1-\frac{1}{4x}}\right)}=\lim_{x\to+\infty}\frac{2x}{2x\left(\sqrt{1+\frac{1}{4x}}+\sqrt{1-\frac{1}{4x}}\right)}$$

$$=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{4x}}+\sqrt{1-\frac{1}{4x}}}=\frac{1}{\sqrt{1+0}+\sqrt{1+0}}=\frac{1}{1+1}=\frac{1}{2}.$$

2. $\lim_{x\to-\infty}(-2x^3+3x^2+x+10)=\lim_{x\to-\infty}(-2x^3)=+\infty$.

3. $\lim_{x\to-2}(3x^2-2x+7)=3(-2)^2-2(-2)+7=12+4+7=23$

4. $f(x)=\dfrac{2x-3}{|2x-3|}$ is not defined at $x=3/2$.

$$|2x-3|=\begin{cases}2x-3 & \text{if } x\geq 3/2\\ -2x+3 & \text{if } x<3/2\end{cases}$$

So $f(x)=\begin{cases}1 & \text{if } x>3/2\\ -1 & \text{if } x<3/2\end{cases}$

$\lim_{x \to 3/2^+} f(x) = 1$ and $\lim_{x \to 3/2^-} f(x) = -1$.

It follows that $\lim_{x \to 3/2} \frac{2x-3}{|2x-3|}$ does not exist.

5. $$\lim_{x \to 0} \left(\frac{1}{x\sqrt{1+x}} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{1-\sqrt{1+x}}{x\sqrt{1+x}} \quad \left(= \tfrac{0}{0} \right)$$

$(1-\sqrt{1+x}$ involves radical$)$

$$= \lim_{x \to 0} \frac{(1-\sqrt{1+x})(1+\sqrt{1+x})}{x\sqrt{1+x}(1+\sqrt{1+x})} = \lim_{x \to 0} \frac{1-(1+x)}{x\sqrt{1+x}(1+\sqrt{1+x})}$$

$$= \lim_{x \to 0} \frac{-x}{x\sqrt{1+x}(1+\sqrt{1+x})} = \lim_{x \to 0} \frac{-1}{\sqrt{1+x}(1+\sqrt{1+x})} = \frac{-1}{1\cdot(1+1)} = -$$

$$\boxed{\lim_{x \to 0} \frac{1}{x\sqrt{1+x}} - \frac{1}{x} = -1/2}$$

6. $$\lim_{x \to 2} \frac{4x-8}{\sqrt{2x+5}-\sqrt{x^2+5}} = \left(\text{“} \tfrac{0}{0} \text{”} \right)$$

$$= \lim_{x \to 2} \frac{(4x-8)(\sqrt{2x+5}+\sqrt{x^2+5})}{(\sqrt{2x+5}+\sqrt{x^2+5})(\sqrt{2x+5}-\sqrt{x^2+5})} = \lim_{x \to 2} \frac{(4x-8)(\sqrt{2x+5}+\sqrt{x^2+5})}{2x+5-(x^2+5)}$$

$$= \lim_{x \to 2} \frac{4(x-2)(\sqrt{2x+5}+\sqrt{x^2+5})}{2x-x^2} = \lim_{x \to 2} \frac{4\cancel{(x-2)}(\sqrt{2x+5}+\sqrt{x^2+5})}{-x\cancel{(x-2)}}$$

$$= \lim_{x \to 2} \frac{4(\sqrt{2x+5}+\sqrt{x^2+5})}{-x} = \frac{4(\sqrt{9}+\sqrt{9})}{-2} = \frac{4\times 6}{-2} = -12$$

7. $\lim\limits_{x \to +\infty} \dfrac{-x^3+2x^2+7x+1}{12+3x-4x^3} = \dfrac{-1}{-4} = \dfrac{1}{4}$.

(rational function with numerator and denominator of the same degree)

8. $\lim\limits_{x \to -2^-} \dfrac{x^2+2x}{x^2-4} \quad \left(= \text{"}\tfrac{0}{0}\text{"}\right)$

$$= \lim_{x \to -2^-} \frac{x(x+2)}{(x-2)(x+2)} = \lim_{x \to -2} \frac{x}{x-2} = \frac{-2}{-2-2} = \frac{2}{4} = \frac{1}{2}$$

P2. $f(x)$ has two breakpoints (at $x=-2$ and $x=3$).

* $\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} \dfrac{1}{x+2} = -\infty$

$\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} (x^2-5) = 4-5 = -1$

The two one-sided limits at 2 are not both equal to a finite number. So, the limit $\lim\limits_{x \to -2} f(x)$ does not exist.

* $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} \sqrt{x+13} = \sqrt{3+13} = \sqrt{16} = 4$

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} x^2-5 = 9-5 = 4$

Both left and right limits at 3 exist and are equal to 4. So $\lim\limits_{x \to 3} f(x) = 4$.

P3. The function $f(x) = \begin{cases} 8-x^2, & x \geq -3 \\ \frac{k}{x}, & x < -3 \end{cases}$

is well defined at $x=-3$, with $g(-3) = 8-9 = -1$

$f(x)$ will be continuous at $x=-3$ iff

$\lim\limits_{x \to -3} f(x)$ exists and is equal to $f(-3) = -1$.

Let $k \in \mathbb{R}$. $\lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^-} (8-x^2) = 8-9 = -1$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \frac{k}{x} = \frac{k}{-3}.$$

The limit exists iff $\lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^+} f(x) \in \mathbb{R}$

$$\Leftrightarrow -1 = \frac{k}{-3} \Leftrightarrow \boxed{k = 3}$$

$k=3$ is the only value that makes $f(x)$ continuous at $x=-3$.

P4. Set $f(x) := x^3 + 4x - 7$. $f(x)$ is <u>continuous</u> everywhere (as a polynomial). By the I.V.T, $f(x)$ has a root on any closed interval where $f(x)$ changes its signs.

x	$\cdots -2$	-1	0	1	2	3	$\cdots$	$+\infty$
$f(x) = x^3+4x-7$	-25	-12	-7	-2	9	32	$\cdots$	
	<0	<0	<0	<0	>0	>0		

$f(x)$ is continuous and changes sign on $[1,2]$.

So, $f(x)$ has a root on $[1,2]$ (a closed interval of length 1)

Divide $[1,2]$ into two subintervals of the same length $1/2$: $[1, \frac{3}{2}]$ and $[3/2, 2]$

One of these two subinterval contains a root:

$f(x)$ is continuous on both closed intervals.
So, $f(x)$ has a root on the interval where it changes sign

$$f(1) = 1+4-7 = -2 < 0$$
$$f(3/2) = \frac{27}{4} + 6 - 7 = \frac{23}{4} > 0$$
$$f(2) = 8+8-7 = 9 > 0$$

$f(x)$ changes sign on $[1, 3/2]$
so, $[1, 3/2]$ contains a root for $f(x)=0$

P5. Set $f(x) = x^3 - 15x + 1$.

$f(x)$ is everywhere continuous. in particular it is continuous on the finite closed interval $[0,4]$ and on any other finite closed interval.

But $\left.\begin{array}{l} f(0) = 0-0+1 = 1 > 0 \\ f(4) = 64-60+1 = 5 > 0 \end{array}\right\} \Rightarrow$ We can conclude nothing

x	$-\infty$	0	1	2	3	4	$+\infty$
$x^3-15x+1$		1	-13	-21	-17	5	
		>0	<0	<0	<0	>0	

$$\left.\begin{array}{l} f(0) = 1 > 0 \\ f(1) = -13 < 0 \\ f(4) = 5 > 0 \end{array}\right\} \Rightarrow [0,1] \text{ and } [1,4] \text{ contain roots of } x^3 - 15x + 1$$