Practice Problems S6

1. Given

$$f(x) = \frac{x^2-4}{x^2-1}, \quad f'(x) = \frac{6x}{(x-1)^2(x+1)^2},$$
$$f''(x) = \frac{-6(3x^2+1)}{(x-1)^3(x+1)^3}, \quad \lim_{x\to-\infty} f(x) = 1 = \lim_{x\to+\infty} f(x),$$
$$\lim_{x\to-1^-} f(x) = -\infty = \lim_{x\to1^+} f(x), \quad \lim_{x\to-1^+} f(x) = +\infty = \lim_{x\to1^-} f(x),$$

sketch the graph of f.

2. If the position S of a particle moving along an s-axis is given as a function of the time t by $S(t) = 2t^3 - 9t^2 + 12t$ for $t > 0$,

   (a) find the velocity, $v(t)$ and acceleration, $a(t)$ of the particle;

   (b) find the average velocity, v_{av} of the particle over the time interval $t_1 = 1$ and $t_2 = 2$.

   (c) find all intervals for $t > 0$ when the particle is speeding up and when it is slowing down. When is it stopped?

3. Find the absolute minimum and the absolute maximum values of the function $f(x) = (x^2 + x)^{\frac{2}{3}}$ on $[-2, 3]$.

4. A cylindrical can, open at the top, is to hold 500 cm^3 of beer. Find the height and the radius that minimize the amount of material needed to manufacture the can.

5. An open box is to be made from a 3-ft by 8-ft rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum volume that the open box can have.

6. Evaluate the following integrals:

   (a) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$;

   (b) $\int \frac{4x}{\sqrt{2x+1}} dx$..

Solutions

2. $s(t) = 2t^3 - 9t^2 + 12t\ ,\ t > 0$
(Position function)

(a) $v(t) = s'(t) = \frac{d}{dt} s(t) = 6t^2 - 18t + 12$
$= 6(t-2)(t-1)$

$a(t) = \frac{d}{dt} v(t) = \frac{d^2}{dt^2} s(t) = 12t - 18$
$= 6(2t-3)$

(b) $v_{av} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(2) - s(1)}{2 - 1} =$

(c) Signs for $v(t)$ and $a(t)$:

t	0		1		3/2		2		$+\infty$
$v(t)$		+	0	−		−	0	+	
$a(t)$		−		−	0	+		+	

The particle speeds up when $v(t) > 0$ and $a(t) > 0$ or
$v(t) < 0$ and $a(t) < 0$
i.e. during the time interval from $t_1 = 1$ to $t_2 = 3/2$
and from $t_3 = 2$.
It slows down otherwise, i.e. from 0 to 1 and from
3/2 to 2.
It is stopped when $v = 0$, i.e. at time $t = 1$ and $t = 2$

3. The function $f(x) = (x^2+x)^{2/3}$ is continuous everywhere, in particular $f(x)$ is continuous on the finite closed interval $[-2,3]$. By the extreme-value theorem, f has an absolute maximum and an absolute minimum on $[-2,3]$

Critical and singular points:

$$f'(x) = \left[(x^2+x)^{2/3}\right]' = \frac{2}{3}(2x+1)(x^2+x)^{-1/3}$$

$$= \frac{2(2x+1)}{3(x^2+x)^{\frac{1}{3}}}$$

So, f has one critical point $x = -\frac{1}{2} \in [-2,3]$
(as $f'(\frac{1}{2}) = 0$)

two singular points

$x = 0$ and $x = -1$
(as f is not diff at these points)

$f(-1) = 0$

$f(-\frac{1}{2}) = (\frac{1}{4})^{2/3}$

$f(0) = 0$

$f(-2) = 2^{2/3}$

$f(3) = 12^{2/3} \to$ largest value

Smallest value of f on $[-2,3]$
0 is the absolute minimum value of f attained at $x=-1$ and $x=0$.

$\Rightarrow f$ has an absolute max at $x=3$.

4. Denote by H and R the height and Radius of the can, respectively. Then the volume V of the can is given by

$$V = 500 = \pi R^2 H.$$

The total area of the can (open at the top) is $S = \pi R^2 + 2\pi R H$.

The problem is to minimize S:

$$500 = \pi R^2 H \Rightarrow H = \frac{500}{\pi R^2}.$$

So, $S \equiv S(R) = \pi R^2 + \frac{1000}{R}$.

S is a function of R with $R \in (0, +\infty)$

We have $\lim_{R \to 0^+} S(R) = +\infty = \lim_{R \to +\infty} S(R)$

$\Rightarrow S(R)$ has an absolute minimum on $(0, +\infty)$

$S(R)$ is diff on $(0, +\infty)$, so it has no singular point on $(0, +\infty)$.

$$S'(R) = 2\pi R + \left(-\frac{1000}{R^2}\right) = \frac{2\pi R^3 - 1000}{R^2}$$

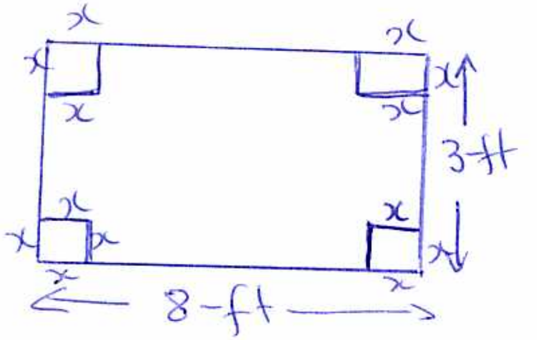
$S'(R) = 0 \Leftrightarrow R = \frac{10}{\sqrt[3]{2\pi}}$ is the only critical point.

Therefore, $R_{min} = \frac{10}{\sqrt[3]{2\pi}}$ is the radius of the can that minimize the cost.

And the height of the can must be

$$H_{min} = \frac{500}{\pi R_{min}^2} = \frac{500}{\pi \frac{100}{\sqrt[3]{4\pi^2}}} = 5\sqrt[3]{\frac{4}{\pi}}$$

5.



Denote by x the side of the squares to be cut from the corners

After bending up the sides, the resulting open box will have a width of $3-2x$ ft

length $8-2x$ ft

heigth x ft.

So it volume is given as a function of x by

$$V(x) = x(3-2x)(8-2x) = 4x^3 - 22x^2 + 24x,$$

where x varies from 0 to $3/2$. So, the problem is to maximize $V(x)$.

$V(x)$ is a polynomial in $x \Rightarrow$ no singular points

$$V'(x) = (4x^3 - 22x^2 + 24x)'$$
$$= 12x^2 - 44x + 24$$
$$= 4(x-3)(2x-3).$$

So $V(x)$ has one critical point $\frac{2}{3}$ that belongs to $[0, \frac{3}{2}]$.

So, the maximum volume that the box can have is

$$V(2/3) = 2/3\,(3 - 4/3)(8 - 4/3) = \frac{200}{27}.$$

6. (a) $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$

Integration by substitution:

Set $u = \sqrt{x}$. $\Rightarrow du = d\sqrt{x} = \frac{dx}{2\sqrt{x}}$.

$\Rightarrow 2du = \frac{dx}{\sqrt{x}}$.

So, $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx = \int \cos u \, du$

$= \sin u + c$

$= \sin(\sqrt{x}) + c$

(b) $\int \frac{4x\, dx}{\sqrt{2x+1}}$

Set $u = \sqrt{2x+1} \Rightarrow du = \frac{2dx}{2\sqrt{2x+1}} = \frac{dx}{\sqrt{2x+1}}$.

$\Downarrow$

$x = (u^2-1)/2$.

So $\int \frac{4x\,dx}{\sqrt{2x+1}} = \int 4x \frac{dx}{\sqrt{2x+1}} = \int 4\frac{(u^2-1)}{2} \cdot du$

$= 2\int (u^2-1)\, du = 2\left(\frac{u^3}{3} - u\right) + c$

$= 2\left(\frac{(\sqrt{2x+1})^3}{3} - \frac{\sqrt{2x-1}}{1}\right) + c.$