ARCHIMEDES AND THE ESTIMATION OF π

Consider a circle of radius 1. Let s_n be the length of a side of the regular polygon with n sides inscribed in the circle, and let p_n be the perimeter of the polygon. Let t_n be the length of a side of the regular polygon with n sides circumscribed about the circle, and P_n the perimeter. With this notation, we have the inequality

$$p_n < 2\pi < P_n.$$

1. Show that s_n and t_n satisfy

$$s_n = 2\sin\frac{\pi}{n} \quad \text{and} \quad t_n = 2\tan\frac{\pi}{n}$$

2. In the lecture, we have already shown that Archimedes found a doubling formula for the inscribed case, and it had the form

$$s_{2n}^2 = 2 - \sqrt{4 - s_n^2}.$$

Using the double angle formula for the tangent (or whatever other method you prefer) show that

$$t_{2n} = \frac{2t_n}{2 + \sqrt{4 + t_n^2}}.$$

3. By using a calculator, writing a script in your favourite language (see the Python script that follows), or using a spreadsheet, use these two recursion formulae to fill in a table of values of $\{n, s_n, t_n, p_n/2, P_n/2\}$ for $n = 6, 12, \ldots, 96$ to see how well you are approximating π.

4. If you were to do the previous calculations by hand, you would very quickly wish that you did not have to extract so many square roots. It is a bit surprising that this can be done. To see how, show that

$$p_{2n} = \sqrt{p_n P_{2n}} \quad \text{and} \quad P_{2n} = \frac{2p_n P_n}{p_n + P_n}$$

Starting with $p_4 = 4\sqrt{2}$ and $P_4 = 8$ for inscribed and circumscribed squares, use these recursive formulas to calculate p_{64} and P_{64}. What bounds on π does this calculation give?

Here is a python script and its output that approximates pi using Archimedes scheme of inscribed and circumscribed polygons.

```
#! /usr/bin/python

from math import *

n, s, t = 4, sqrt(2), 2

for k in range(1, 11):
    lowpi, highpi = n*s/2, n*t/2
    print n, lowpi, highpi
    s = sqrt(2 - sqrt(4 - s*s))
    t = 2 * t / (2 + sqrt(4 + t*t))
    n = 2 * n
```

The output of the script:

```
bates@tiki:~/math/courses/math251$ ./pi.py

   4  2.82842712475 4
   8  3.06146745892 3.31370849898
  16  3.12144515226 3.18259787807
  32  3.13654849055 3.15172490743
  64  3.14033115695 3.14411838525
 128  3.14127725093 3.14222362994
 256  3.14151380114 3.14175036917
 512  3.14157294037 3.1416320807
1024  3.14158772528 3.14160251026
2048  3.1415914215  3.14159511775
```