QUADRATURE OF THE PARABOLA

In this exercise you will recover a beautiful theorem on the parabola due to the greatest mathematician of antiquity, ARCHIMEDES.

Consider the parabola P given by $y = x^2$ and the line L given by $y = 2x + 8$.

1. Let $A = (a, a^2)$ and $B = (b, b^2)$ with $a < b$ be the intersection points of L and P. Find A and B.
2. Let C be a point on the parabola P between A and B. Find C so that the area of the triangle $\triangle ABC$ is maximized. Hint: what can you say about the line tangent to the parabola P at C?
3. What is the area of $\triangle ABC$?
4. Let R be the point on the parabola P between A and C so that the area of $\triangle ACR$ is a maximum, and let S be the point on the parabola P between B and C so that the area of $\triangle BCS$ is a maximum. Find the area of these two new triangles.
5. What is the ratio of areas
$$\text{area}(\triangle ABC) : \text{area}(\triangle ACR) : \text{area}(\triangle BCS)?$$
6. Draw an accurate picture of L, P and the three triangles.
7. By summing the appropriate geometric series, find the area of the region enclosed by the chord L and the parabola P. For those who already know integration, also compute this area using an integral.
8. (Extra for experts). Redo the exercise using the line $y = mx + d$. You may assume that d and m are both positive.