1. Answer the following true, false or numerically.

   (a) The composition of two odd functions is an even function.

   (b) If $f(x) = 3x^6 + 4x^4 + 2x^2$, then f is concave up on the whole real line.

   (c) The graph of $y = (x^2 - x - 6)/(x - 3)$ has a vertical asymptote at $x = 3$.

   (d) Evaluate the integral

   $$\int_5^8 \sqrt{3x+1}\,dx.$$

   (e) Evaluate the integral

   $$\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx.$$

   (f) Find $G'(3)$ if

   $$G(x) = \int_1^x x^2\sqrt{1+u}\,du.$$

   (g) Find the interval on which the graph of $f(x)$ is concave up if

   $$f(x) = \int_0^x \frac{1+t}{1+t^2}\,dt$$

   (h) What constant acceleration will cause a car to increase its velocity from 45 to 60 kilometers per hour in 10 seconds?

   (i) Compute the limit

   $$\lim_{x\to\infty} \frac{(\ln 2x)^2}{3x}$$

2. A fly is crawling from left to right along the top of the curve $y = 7 - x^2$. A spider waits at the point $(4, 0)$. Find the distance between the fly and the spider when they first see each other.

3. Let $P(a, b)$ be a point on the first quadrant portion of the curve $y = 1/x$ and let the tangent line at P intersect the x-axis at A. Show that the triangle AOP is isosceles and determine its area.

4. Define $\sinh y = (e^y - e^{-y})/2$, and $\cosh y = (\sinh y)'$. Show that $\sinh^2 y + \cosh^2 y = 1$.

5. Compute the third order Taylor polynomial to the square root function $y = \sqrt{x}$ and use it to estimate $\sqrt{13}$. Hint: what point should you expand about?

6. The sum of two nonnegative numbers is 36. Find the numbers if the sum of one number and the square root of the other is to be as large as possible.

7. Find the equation of the line that is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the first quadrant and that forms with the cooordinate axes the triangle with smallest area. (This problem is harder than the others, so may only be doable by 'A' students.)

8. Sketch the graph of $f(x) = x^3 - x + 1$. Be sure to label all inflection points, asymptotes, regions of increase and decrease, critical points, and x and y intercepts.

9. Determine a and b so that $f(x) = a\sqrt{x} + b/\sqrt{x}$ has the point $(4, 13)$ as an inflection point.

10. Starting at 4 am, a monk slowly climbed to the top of a mountain, arriving at noon. The next day, he returned along the same path, starting at 5 am and getting to the bottom at 11 am. Show that at some point along the path, his watch showed the same time on both days.

11. On a clock face let N be the point where the 9 is located, O the center of the clock face, and T the tip of the minute hand. If we assume the distance $|ON| = |OT|$, how fast is the angle $\angle NTO$ changing when the time is five minutes past one?

12. A flower bed will be in the shape of a sector of a circle (a pie-shaped region) of radius r and vertex angle ϕ. Find r and ϕ if its area is a constant A and the perimeter is a minimum.

13. Find the equations of the two tangent lines to the circle

$$x^2 + 4x + y^2 + 3 = 0$$

that pass through the origin.

14. The hour and minute hands of a clock are six and eight centimeters long, respectively. How fast are the tips of the hands separating at 12:20?

15. Of all right circular cylinders with a given surface area find the one with maximum volume. Note: The ends of the cylinders are closed.

16. From what height above the earth must a ball be dropped in order to strike the ground with a speed of 45 meters per second? Recall that the acceleration due to gravity is 10 m/s^2.

17. Let $ad - bc = 1$ and
$$y = f(x) = \frac{ax+b}{cx+d}.$$
Find the inverse function $x = f^{-1}(y)$ and compute its derivative at $y = 1$

18. Recall that the natural logarithm of a positive number x may be defined as
$$\ln x = \int_1^x \frac{1}{t}\,dt.$$
Show that

    (a)
    $$\int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt$$
    and hence conclude that $\ln(ab) = \ln a + \ln b$.

    (b) Prove the addition law for the exponential function
    $$e^{a+b} = e^a e^b.$$
    Hint: you can use the result of the previous section and the fact that the natural logarithm and the exponential function are inverse functions.

19. Prove that if f is an even function and g is an odd function, then the product function fg is an odd function.

20. Give an example to show that an inflection point can be a critical point.

21. Compute the limit

$$\lim_{u \to 1} \frac{u^4 - 1}{u^3 - 1}$$

22. Compute the limit

$$\lim_{x \to -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$$

23. If $f(x) = 1 - 1/x$, and $g(x) = 1/(1 - x)$, compute the derivative of $f \circ g(x)$ when $x = -1$.

24. Find the y coordinates of the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$.

25. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis. Are the tangents to the curve at these points parallel?

26. The strength S of a rectangular beam is proportional to its width times the square of its depth. Find the dimensions of the strongest beam that can be cut from a 12 inch cylindrical log.

27. Show that $\cos x = x$ has at least one solution. For more bonus marks, prove that there is precisely one solution.

28. Prove that $(x^n)' = nx^{n-1}$ for $n = 1, 2, 3, \ldots$.

29. Sketch the graph of

$$y = \frac{x^3 + 1}{x}$$

Be sure to label all asymptotes, critical points and inflection points. Indicate all regions where the function is increasing, decreasing, concave up and concave down.

30. Give precise definitions or statements of the following

    (a) The Chain Rule.
    (b) A critical point of a function.
    (c) The Mean Value Theorem
    (d) The Fundamental Theorem of the Calculus.

31. For a differentiable function f on the interval $[a, b]$, the mean value theorem says that there is at least one point $\xi \in (a, b)$ such that $f'(\xi)(b-a) = f(b) - f(a)$. Show that if $f(x) = x^2$, then the point ξ guaranteed by the theorem is the arithmetic mean of the end points of the interval. That is, $\xi = (a+b)/2$.

32. Plot the graph of $y = x^4 - 4x^3 + 10$. Label all relevant qualitative features.

33. Find all the asymptotes of the graph of $f(x) = (x^2 - 3)/(2x - 4)$.

34. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

35. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass while the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.

36. You operate a tour service that offers the following rates:

    (a) \$200 per person of 50 people (the minimum number to book the tour) go on the tour.

    (b) For each additional person, up to a maximum of 80 people total, everyone's charge is reduced by \$2.

    It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit?

37. You are driving along the highway at 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft?

38. Sketch the parabolic arch $y = h - (4h/b^2)x^2$ for $-b/2 \le x \le b/2$, assuming that h and b are positive. Find the area enclosed by the arch and the x-axis. This is a theorem of Archimedes.

39. Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)\, dt.$$

Give reasons for your answers.

(a) h is a twice differentiable function of x.

(b) h and h' are both continuous.

(c) The graph of h has a horizontal tangent at $x = 1$.

(d) h has a local maximum at $x = 1$.

(e) h has a local minimum at $x = 1$.

(f) The graph of h has an inflection point at $x = 1$.

(g) The graph of h' crosses the x-axis at $x = 1$.

40. Find $f(4)$ if

(a)

$$\int_0^{x^2} f(t)\, dt = x \cos \pi x,$$

(b)

$$\int_0^{f(x)} t^2\, dt = x \cos \pi x.$$

41. A rectangle has vertices at $(0, 0)$, $(x, 0)$, $(0, y)$ and (x, y) where $y > 0$ and $y = (\ln x)/x^2$. What values of x and y give the largest value of xy? What is the area of this rectangle?

42. Suppose that a right circular cylinder of radius r and height h is inscribed in a right circular cone of radius R and height H. Find the value of r (in terms of R and H) that maximizes the total surface area of the cylinder (including the top and bottom). As you will see, the solution depends on whether $H \leq 2R$ or $H \geq 2R$.