APPENDIX A

Real Numbers, Intervals, and Inequalities

REAL NUMBERS

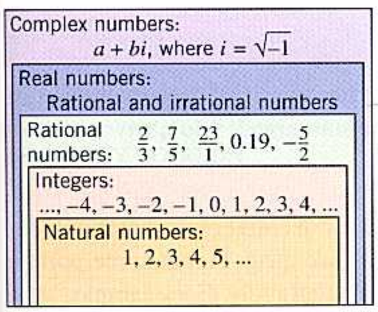


Figure A.1

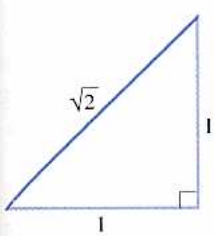


Figure A.2

Figure A.1 describes the various categories of numbers that we will encounter in this text. The simplest numbers are the ***natural numbers***

$$1,\ 2,\ 3,\ 4,\ 5, \ldots$$

These are a subset of the ***integers***

$$\ldots,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4, \ldots$$

and these in turn are a subset of the ***rational numbers***, which are the numbers formed by taking ratios of integers (avoiding division by 0). Some examples are

$$\tfrac{2}{3},\quad \tfrac{7}{5},\quad 23 = \tfrac{23}{1},\quad 0.19 = \tfrac{19}{100},\quad -\tfrac{5}{2} = \tfrac{-5}{2} = \tfrac{5}{-2}$$

The early Greeks believed that every measurable quantity had to be a rational number. However, this idea was overturned in the fifth century B.C. by Hippasus of Metapontum* who demonstrated the existence of ***irrational numbers***, that is, numbers that cannot be expressed as the ratio of two integers. Using geometric methods, he showed that the length of the hypotenuse of the triangle in Figure A.2 could not be expressed as a ratio of integers, thereby proving that $\sqrt{2}$ is an irrational number. Some other examples of irrational numbers are

$$\sqrt{3},\quad \sqrt{5},\quad 1+\sqrt{2},\quad \sqrt[3]{7},\quad \pi,\quad \cos 19^\circ$$

The rational and irrational numbers together comprise what is called the ***real number system***, and both the rational and irrational numbers are called ***real numbers***.

COMPLEX NUMBERS

Because the square of a real number cannot be negative, the equation

$$x^2 = -1$$

has no solutions in the real number system. In the eighteenth century mathematicians remedied this problem by inventing a new number, which they denoted by

$$i = \sqrt{-1}$$

and which they defined to have the property $i^2 = -1$. This, in turn, led to the development

* HIPPASUS OF METAPONTUM (circa 500 B.C.). A Greek Pythagorean philosopher. According to legend, Hippasus made his discovery at sea and was thrown overboard by fanatic Pythagoreans because his result contradicted their doctrine. The discovery of Hippasus is one of the most fundamental in the entire history of science.

that a is not a member of the set A we will write $a \notin A$ (read "a does not belong to A"). For example, if A is the set of positive integers, then $5 \in A$, but $-5 \notin A$. Sometimes sets arise that have no members (e.g., the set of odd integers that are divisible by 2). A set with no members is called an ***empty set*** or a ***null set*** and is denoted by the symbol $\varnothing$.

Some sets can be described by listing their members between braces. The order in which the members are listed does not matter, so, for example, the set A of positive integers that are less than 6 can be expressed as

$$A = \{1, 2, 3, 4, 5\} \quad \text{or} \quad A = \{2, 3, 1, 5, 4\}$$

We can also write A in *set-builder notation* as

$$A = \{x : x \text{ is an integer and } 0 < x < 6\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 6$." In general, to express a set S in set-builder notation we write $S = \{x : \underline{\qquad}\}$ in which the line is replaced by a property that identifies exactly those elements in the set S.

If every member of a set A is also a member of a set B, then we say that A is a ***subset*** of B and write $A \subseteq B$. For example, if A is the set of positive integers and B is the set of all integers, then $A \subseteq B$. If two sets A and B have the same members (i.e., $A \subseteq B$ and $B \subseteq A$), then we say that A and B are ***equal*** and write $A = B$.

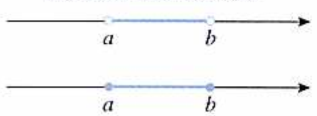
INTERVALS

In calculus we will be concerned with sets of real numbers, called ***intervals***, that correspond to line segments on a coordinate line. For example, if $a < b$, then the ***open interval*** from a to b, denoted by (a, b), is the line segment extending from a to b, *excluding* the endpoints; and the ***closed interval*** from a to b, denoted by $[a, b]$, is the line segment extending from a to b, *including* the endpoints (Figure A.6). These sets can be expressed in set-builder notation as

$$(a, b) = \{x : a < x < b\} \qquad \text{The open interval from } a \text{ to } b$$

$$[a, b] = \{x : a \le x \le b\} \qquad \text{The closed interval from } a \text{ to } b$$

The open interval (a, b)



The closed interval $[a, b]$

Figure A.6

REMARK. Observe that in this notation and in the corresponding Figure A.6, parentheses and open dots mark endpoints that are excluded from the interval, whereas brackets and closed dots mark endpoints that are included in the interval. Observe also that in set-builder notation for the intervals, it is understood that x is a real number, even though it is not stated explicitly.

As shown in Table 1, an interval can include one endpoint and not the other; such intervals are called ***half-open*** (or sometimes ***half-closed***). Moreover, the table also shows that it is possible for an interval to extend indefinitely in one or both directions. To indicate that an interval extends indefinitely in the positive direction we write $+\infty$ (read "positive infinity") in place of a right endpoint, and to indicate that an interval extends indefinitely in the negative direction we write $-\infty$ (read "negative infinity") in place of a left endpoint. Intervals that extend between two real numbers are called ***finite intervals***, whereas intervals that extend indefinitely in one or both directions are called ***infinite intervals***.

REMARK. By convention, infinite intervals of the form $[a, +\infty)$ or $(-\infty, b]$ are considered to be closed because they contain their endpoint, and intervals of the form $(a, +\infty)$ and $(-\infty, b)$ are considered to be open because they do not include their endpoint. The interval $(-\infty, +\infty)$, which is the set of all real numbers, has no endpoints and can be regarded as either open or closed, as convenient. This set is often denoted by the special symbol $\mathbb{R}$. To distinguish verbally between the open interval $(0, +\infty) = \{x : x > 0\}$ and the closed interval $[0, +\infty) = \{x : x \ge 0\}$, we will call x ***positive*** if $x > 0$ and ***nonnegative*** if $x \ge 0$. Thus, a positive number must be nonnegative, but a nonnegative number need not be positive, since it might possibly be 0.

Table 1

INTERVAL NOTATION	SET NOTATION	GEOMETRIC PICTURE	CLASSIFICATION
(a, b)	$\{x : a < x < b\}$		Finite; open
$[a, b]$	$\{x : a \le x \le b\}$		Finite; closed
$[a, b)$	$\{x : a \le x < b\}$		Finite; half-open
$(a, b]$	$\{x : a < x \le b\}$		Finite; half-open
$(-\infty, b]$	$\{x : x \le b\}$		Infinite; closed
$(-\infty, b)$	$\{x : x < b\}$		Infinite; open
$[a, +\infty)$	$\{x : x \ge a\}$		Infinite; closed
$(a, +\infty)$	$\{x : x > a\}$		Infinite; open
$(-\infty, +\infty)$	$\mathbb{R}$		Infinite; open and closed

UNIONS AND INTERSECTIONS OF INTERVALS

If A and B are sets, then the ***union*** of A and B (denoted by $A \cup B$) is the set whose members belong to A or B (or both), and the ***intersection*** of A and B (denoted by $A \cap B$) is the set whose members belong to both A and B. For example,

$$\{x : 0 < x < 5\} \cup \{x : 1 < x < 7\} = \{x : 0 < x < 7\}$$
$$\{x : x < 1\} \cap \{x : x \ge 0\} = \{x : 0 \le x < 1\}$$
$$\{x : x < 0\} \cap \{x : x > 0\} = \varnothing$$

or in interval notation,

$$(0, 5) \cup (1, 7) = (0, 7)$$
$$(-\infty, 1) \cap [0, +\infty) = [0, 1)$$
$$(-\infty, 0) \cap (0, +\infty) = \varnothing$$

ALGEBRAIC PROPERTIES OF INEQUALITIES

The following algebraic properties of inequalities will be used frequently in this text. We omit the proofs.

> **A.1** THEOREM (***Properties of Inequalities***). *Let a, b, c, and d be real numbers.*
>
> (*a*) *If $a < b$ and $b < c$, then $a < c$.*
>
> (*b*) *If $a < b$, then $a + c < b + c$ and $a - c < b - c$.*
>
> (*c*) *If $a < b$, then $ac < bc$ when c is positive and $ac > bc$ when c is negative.*
>
> (*d*) *If $a < b$ and $c < d$, then $a + c < b + d$.*
>
> (*e*) *If a and b are both positive or both negative and $a < b$, then $1/a > 1/b$.*

If we call the direction of an inequality its *sense*, then these properties can be paraphrased as follows:

(*b*) *The sense of an inequality is unchanged if the same number is added to or subtracted from both sides.*

(*c*) *The sense of an inequality is unchanged if both sides are multiplied by the same positive number, but the sense is reversed if both sides are multiplied by the same negative number.*

(*d*) *Inequalities with the same sense can be added.*

(*e*) *If both sides of an inequality have the same sign, then the sense of the inequality is reversed by taking the reciprocal of each side.*

REMARK. These properties remain true if the symbols $<$ and $>$ are replaced by $\leq$ and $\geq$ in Theorem A.1.

Example 1

STARTING INEQUALITY	OPERATION	RESULTING INEQUALITY
$-2 < 6$	Add 7 to both sides.	$5 < 13$
$-2 < 6$	Subtract 8 from both sides.	$-10 < -2$
$-2 < 6$	Multiply both sides by 3.	$-6 < 18$
$-2 < 6$	Multiply both sides by -3.	$6 > -18$
$3 < 7$	Multiply both sides by 4.	$12 < 28$
$3 < 7$	Multiply both sides by -4.	$-12 > -28$
$3 < 7$	Take reciprocals of both sides.	$\frac{1}{3} > \frac{1}{7}$
$-8 < -6$	Take reciprocals of both sides.	$-\frac{1}{8} > -\frac{1}{6}$
$4 < 5, -7 < 8$	Add corresponding sides.	$-3 < 13$

◀

SOLVING INEQUALITIES

A ***solution*** of an inequality in an unknown x is a value for x that makes the inequality a true statement. For example, $x = 1$ is a solution of the inequality $x < 5$, but $x = 7$ is not. The set of all solutions of an inequality is called its ***solution set***. It can be shown that if one does not multiply both sides of an inequality by zero or an expression involving an unknown, then the operations in Theorem A.1 will not change the solution set of the inequality. The process of finding the solution set of an inequality is called ***solving*** the inequality.

Example 2 Solve $3 + 7x \leq 2x - 9$.

Solution. We will use the operations of Theorem A.1 to isolate x on one side of the inequality.

$3 + 7x \leq 2x - 9$	Given.
$7x \leq 2x - 12$	We subtracted 3 from both sides.
$5x \leq -12$	We subtracted $2x$ from both sides.
$x \leq -\frac{12}{5}$	We multiplied both sides by $\frac{1}{5}$.

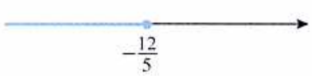


Figure A.7

Because we have not multiplied by any expressions involving the unknown x, the last inequality has the same solution set as the first. Thus, the solution set is the interval $\left(-\infty, -\frac{12}{5}\right]$ shown in Figure A.7. ◀

Example 3 Solve $7 \leq 2 - 5x < 9$.

Solution. The given inequality is actually a combination of the two inequalities

$$7 \leq 2 - 5x \quad \text{and} \quad 2 - 5x < 9$$

We could solve the two inequalities separately, then determine the values of x that satisfy both by taking the intersection of the two solution sets. However, it is possible to work with the combined inequalities in this problem:

$7 \le 2 - 5x < 9$	Given.
$5 \le -5x < 7$	We subtracted 2 from each member.
$-1 \ge x > -\frac{7}{5}$	We multiplied by $-\frac{1}{5}$ and reversed the sense of the inequalities.
$-\frac{7}{5} < x \le -1$	For clarity, we rewrote the inequalities with the smaller number on the left.

Figure A.8

Thus, the solution set is the interval $\left(-\frac{7}{5}, -1\right]$ shown in Figure A.8. ◀

Example 4 Solve $x^2 - 3x > 10$.

Solution. By subtracting 10 from both sides, the inequality can be rewritten as

$$x^2 - 3x - 10 > 0$$

Factoring the left side yields

$$(x+2)(x-5) > 0$$

The values of x for which $x + 2 = 0$ or $x - 5 = 0$ are $x = -2$ and $x = 5$. These values divide the coordinate line into three open intervals,

$$(-\infty, -2), \quad (-2, 5), \quad (5, +\infty)$$

on each of which the product $(x+2)(x-5)$ has constant sign. To determine those signs we will choose an *arbitrary* number in each interval at which we will determine the sign; these are called ***test values***. As shown in Figure A.9, we will use -3, 0, and 6 as our test values. The results can be organized as follows:

INTERVAL	TEST VALUE	SIGN OF $(x+2)(x-5)$ AT THE TEST VALUE
$(-\infty, -2)$	-3	$(-)(-) = +$
$(-2, 5)$	0	$(+)(-) = -$
$(5, +\infty)$	6	$(+)(+) = +$

The pattern of signs in the intervals is shown on the number line in the middle of Figure A.9. We deduce that the solution set is $(-\infty, -2) \cup (5, +\infty)$, which is shown at the bottom of Figure A.9. ◀

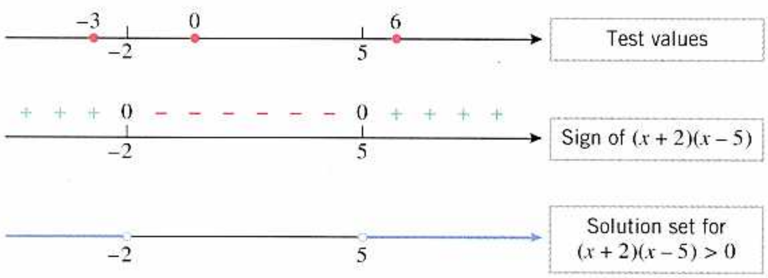


Figure A.9

Example 5 Solve $\dfrac{2x-5}{x-2} < 1$.

Solution. We could start by multiplying both sides by $x - 2$ to eliminate the fraction. However, this would require us to consider the cases $x - 2 > 0$ and $x - 2 < 0$ separately because the sense of the inequality would be reversed in the second case, but not the first. The following approach is simpler:

$$\frac{2x-5}{x-2} < 1 \quad \text{Given.}$$

$$\frac{2x-5}{x-2} - 1 < 0 \quad \text{We subtracted 1 from both sides to obtain a 0 on the right.}$$

$$\frac{(2x-5)-(x-2)}{x-2} < 0 \quad \text{We combined terms.}$$

$$\frac{x-3}{x-2} < 0 \quad \text{We simplified.}$$

The quantity $x - 3$ is zero if $x = 3$, and the quantity $x - 2$ is zero if $x = 2$. These values divide the coordinate line into three open intervals,

$$(-\infty, 2), \quad (2, 3), \quad (3, +\infty)$$

on each of which the quotient $(x-3)/(x-2)$ has constant sign. Using 0, 2.5, and 4 as test values (Figure A.10), we obtain the following results:

INTERVAL	TEST VALUE	SIGN OF $(x-3)/(x-2)$ AT THE TEST VALUE
$(-\infty, 2)$	0	$(-)/(-) = +$
$(2, 3)$	2.5	$(-)/(+) = -$
$(3, +\infty)$	4	$(+)/(+) = +$

The signs of the quotient are shown in the middle of Figure A.10. From the figure we see that the solution set consists of all real values of x such that $2 < x < 3$. This is the interval $(2, 3)$ shown at the bottom of Figure A.10. ◀

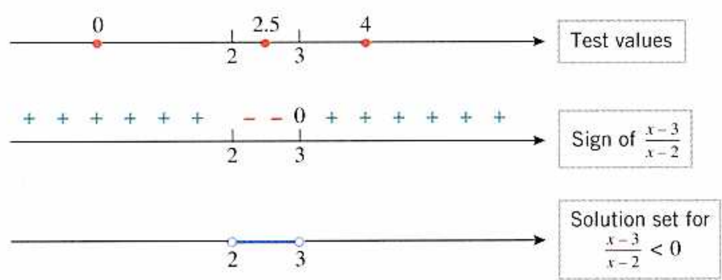


Figure A.10

EXERCISE SET A

1. Among the terms *integer*, *rational*, and *irrational*, which ones apply to the given number?
   (a) $-\frac{3}{4}$ (b) 0 (c) $\frac{24}{8}$
   (d) 0.25 (e) $-\sqrt{16}$ (f) $2^{1/2}$
   (g) 0.020202... (h) 7.000...

2. Which of the terms *integer*, *rational*, and *irrational* apply to the given number?
   (a) 0.31311311131111... (b) 0.729999...
   (c) 0.376237623762... (d) $17\frac{4}{5}$

3. The repeating decimal 0.137137137... can be expressed as a ratio of integers by writing

$$x = 0.137137137\ldots$$
$$1000x = 137.137137137\ldots$$

and subtracting to obtain $999x = 137$ or $x = \frac{137}{999}$. Use this idea, where needed, to express the following decimals as ratios of integers.
   (a) 0.123123123... (b) 12.7777...
   (c) 38.07818181... (d) 0.4296000...

4. Show that the repeating decimal 0.99999... represents the number 1. Since 1.000... is also a decimal representation of 1, this problem shows that a real number can have two different decimal representations. [*Hint:* Use the technique of Exercise 3.]

5. The Rhind Papyrus, which is a fragment of Egyptian mathematical writing from about 1650 B.C., is one of the oldest known examples of written mathematics. It is stated in the papyrus that the area A of a circle is related to its diameter D by

$$A = \left(\tfrac{8}{9}D\right)^2$$

(a) What approximation to π were the Egyptians using?
(b) Use a calculating utility to determine if this approximation is better or worse than the approximation $\frac{22}{7}$.

6. The following are all famous approximations to π:

$$\frac{333}{106}$$ Adrian Athoniszoon, c. 1583

$$\frac{355}{113}$$ Tsu Chung-Chi and others

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right)$$ Ramanujan

$$\frac{22}{7}$$ Archimedes

$$\frac{223}{71}$$ Archimedes

(a) Use a calculating utility to order these approximations according to size.
(b) Which of these approximations is closest to but larger than π?
(c) Which of these approximations is closest to but smaller than π?
(d) Which of these approximations is most accurate?

7. In each line of the accompanying table, check the blocks, if any, that describe a valid relationship between the real numbers a and b. The first line is already completed as an illustration.

a	b	$a<b$	$a\le b$	$a>b$	$a\ge b$	$a=b$
1	6	✓	✓			
6	1					
−3	5					
5	−3					
−4	−4					
0.25	$\frac{1}{3}$					
$-\frac{1}{4}$	$-\frac{3}{4}$					

Table Ex-7

8. In each line of the accompanying table, check the blocks, if any, that describe a valid relationship between the real numbers a, b, and c.

a	b	c	$a<b<c$	$a\le b\le c$	$a<b\le c$	$a\le b<c$
−1	0	2				
2	4	−3				
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$				
−5	−5	−5				
0.75	1.25	1.25				

Table Ex-8

9. Which of the following are always correct if $a \le b$?
(a) $a-3 \le b-3$ (b) $-a \le -b$
(c) $3-a \le 3-b$ (d) $6a \le 6b$
(e) $a^2 \le ab$ (f) $a^3 \le a^2b$

10. Which of the following are always correct if $a \le b$ and $c \le d$?
(a) $a+2c \le b+2d$ (b) $a-2c \le b-2d$
(c) $a-2c \ge b-2d$

11. For what values of a are the following inequalities valid?
(a) $a \le a$ (b) $a < a$

12. If $a \le b$ and $b \le a$, what can you say about a and b?

13. (a) If $a < b$ is true, does it follow that $a \le b$ must also be true?
(b) If $a \le b$ is true, does it follow that $a < b$ must also be true?

14. In each part, list the elements in the set.
(a) $\{x : x^2 - 5x = 0\}$
(b) $\{x : x$ is an integer satisfying $-2 < x < 3\}$

15. In each part, express the set in the notation $\{x : \_\_\_\_\_\_\}$.
(a) $\{1, 3, 5, 7, 9, \ldots\}$
(b) the set of even integers
(c) the set of irrational numbers
(d) $\{7, 8, 9, 10\}$

16. Let $A = \{1, 2, 3\}$. Which of the following sets are equal to A?
(a) $\{0, 1, 2, 3\}$ (b) $\{3, 2, 1\}$
(c) $\{x : (x-3)(x^2-3x+2) = 0\}$

17. In the accompanying figure, let

S = the set of points inside the square
T = the set of points inside the triangle
C = the set of points inside the circle

and let a, b, and c be the points shown. Answer the following as true or false.

(a) $T \subseteq C$ (b) $T \subseteq S$
(c) $a \notin T$ (d) $a \notin S$
(e) $b \in T$ and $b \in C$ (f) $a \in C$ or $a \in T$
(g) $c \in T$ and $c \notin C$

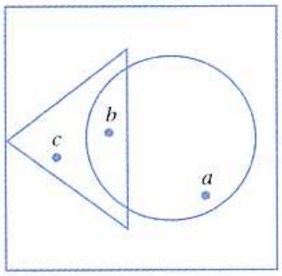


Figure Ex-17

18. List all subsets of
(a) $\{a_1, a_2, a_3\}$ (b) $\varnothing$.

19. In each part, sketch on a coordinate line all values of x that satisfy the stated condition.
(a) $x \leq 4$ (b) $x \geq -3$ (c) $-1 \leq x \leq 7$
(d) $x^2 = 9$ (e) $x^2 \leq 9$ (f) $x^2 \geq 9$

20. In parts (a)–(d), sketch on a coordinate line all values of x, if any, that satisfy the stated conditions.
(a) $x > 4$ and $x \leq 8$
(b) $x \leq 2$ or $x \geq 5$
(c) $x > -2$ and $x \geq 3$
(d) $x \leq 5$ and $x > 7$

21. Express in interval notation.
(a) $\{x : x^2 \leq 4\}$ (b) $\{x : x^2 > 4\}$

22. In each part, sketch the set on a coordinate line.
(a) $[-3, 2] \cup [1, 4]$ (b) $[4, 6] \cup [8, 11]$
(c) $(-4, 0) \cup (-5, 1)$ (d) $[2, 4) \cup (4, 7)$
(e) $(-2, 4) \cap (0, 5]$ (f) $[1, 2.3) \cup (1.4, \sqrt{2})$
(g) $(-\infty, -1) \cup (-3, +\infty)$ (h) $(-\infty, 5) \cap [0, +\infty)$

In Exercises 23–44, solve the inequality and sketch the solution on a coordinate line.

23. $3x - 2 < 8$
24. $\frac{1}{5}x + 6 \geq 14$
25. $4 + 5x \leq 3x - 7$
26. $2x - 1 > 11x + 9$
27. $3 \leq 4 - 2x < 7$
28. $-2 \geq 3 - 8x \geq -11$
29. $\dfrac{x}{x-3} < 4$
30. $\dfrac{x}{8-x} \geq -2$
31. $\dfrac{3x+1}{x-2} < 1$
32. $\dfrac{\frac{1}{2}x - 3}{4+x} > 1$
33. $\dfrac{4}{2-x} \leq 1$
34. $\dfrac{3}{x-5} \leq 2$
35. $x^2 > 9$
36. $x^2 \leq 5$
37. $(x-4)(x+2) > 0$
38. $(x-3)(x+4) < 0$
39. $x^2 - 9x + 20 \leq 0$
40. $2 - 3x + x^2 \geq 0$
41. $\dfrac{2}{x} < \dfrac{3}{x-4}$
42. $\dfrac{1}{x+1} \geq \dfrac{3}{x-2}$
43. $x^3 - x^2 - x - 2 > 0$
44. $x^3 - 3x + 2 \leq 0$

In Exercises 45 and 46, find all values of x for which the given expression yields a real number.

45. $\sqrt{x^2 + x - 6}$
46. $\sqrt{\dfrac{x+2}{x-1}}$

47. Fahrenheit and Celsius temperatures are related by the formula $C = \frac{5}{9}(F - 32)$. If the temperature in degrees Celsius ranges over the interval $25 \leq C \leq 40$ on a certain day, what is the temperature range in degrees Fahrenheit that day?

48. Every integer is either even or odd. The even integers are those that are divisible by 2, so n is even if and only if $n = 2k$ for some integer k. Each odd integer is one unit larger than an even integer, so n is odd if and only if $n = 2k + 1$ for some integer k. Show:
(a) If n is even, then so is n^2
(b) If n is odd, then so is n^2.

49. Prove the following results about sums of rational and irrational numbers:
(a) rational + rational = rational
(b) rational + irrational = irrational.

50. Prove the following results about products of rational and irrational numbers:
(a) rational · rational = rational
(b) rational · irrational = irrational (provided the rational factor is nonzero).

51. Show that the sum or product of two irrational numbers can be rational or irrational.

52. Classify the following as rational or irrational and justify your conclusion.
(a) $3 + \pi$ (b) $\frac{3}{4}\sqrt{2}$
(c) $\sqrt{8}\sqrt{2}$ (d) $\sqrt{\pi}$
(See Exercises 49 and 50.)

53. Prove: The average of two rational numbers is a rational number, but the average of two irrational numbers can be rational or irrational.

54. Can a rational number satisfy $10^x = 3$?

55. Solve: $8x^3 - 4x^2 - 2x + 1 < 0$.

56. Solve: $12x^3 - 20x^2 \geq -11x + 2$.

57. Prove: If a, b, c, and d are positive numbers such that $a < b$ and $c < d$, then $ac < bd$. (This result gives conditions under which inequalities can be "multiplied together.")

58. Is the number represented by the decimal

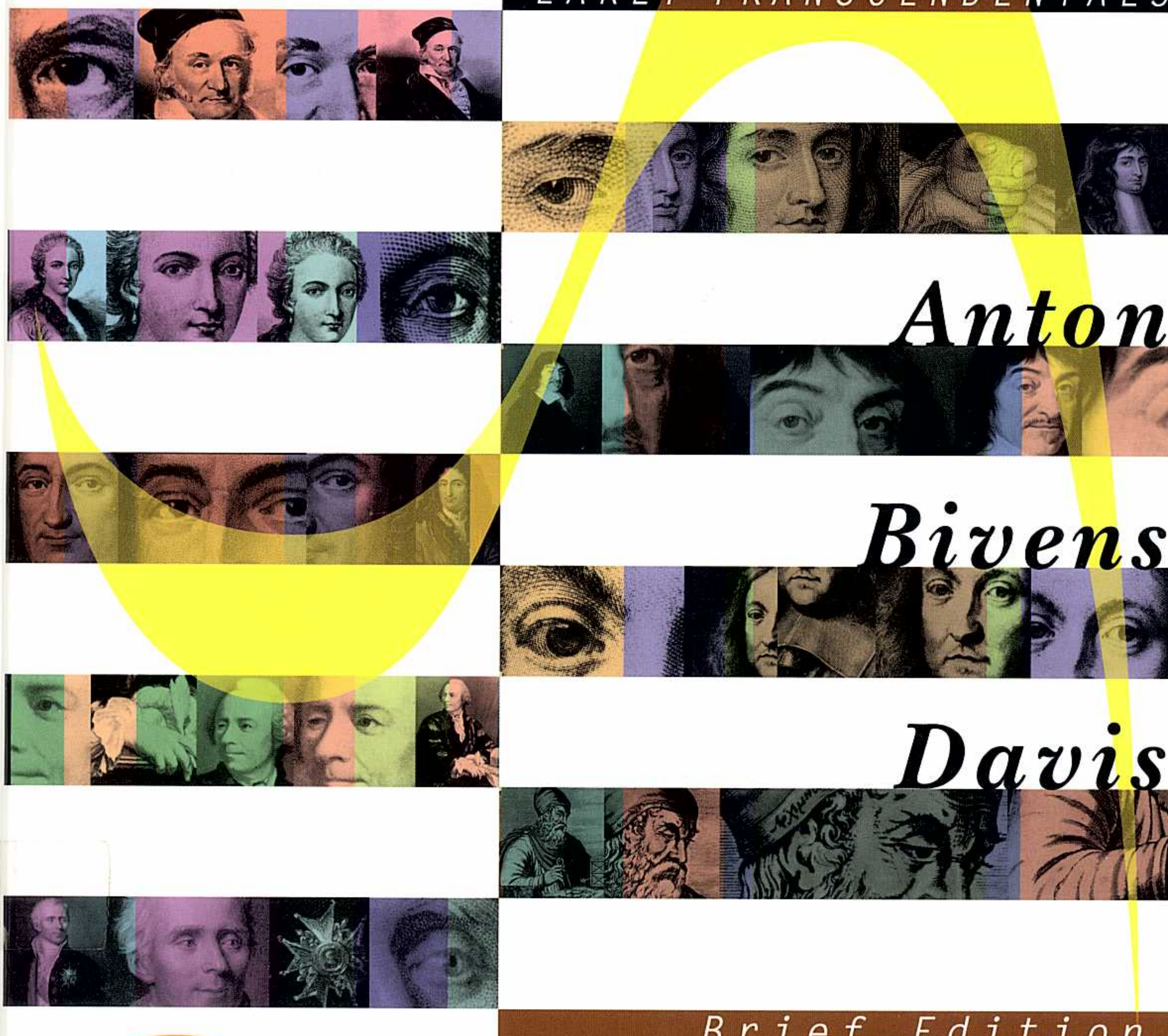
$$0.101001000100001000001\ldots$$

rational or irrational? Explain your reasoning.

7TH *edition*

CALCULUS

EARLY TRANSCENDENTALS



Anton

Bivens

Davis

Brief Edition

CD *inside*