THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 251 - L08

Midterm Examination

Fall 2003

SOLNS

Thursday, October 30, 2003

Duration: Approx. 1 hour and 10 minutes

Show your work.

Total Points = 40

Calculators or notes are not permitted.

Good Luck.

SURNAME	GIVEN NAMES

SIGNATURE	ID NUMBER

[18] 1. (a) If $y = \dfrac{x-1}{2x^2+3}$ then $\dfrac{dx}{dy}|_{x=1} =$ $\frac{1}{5}$.

(b) If $xy = 4$ then $\dfrac{dy}{dx}|_{x=1} =$ -4 .

(c) If $y = \sec(x^2)$ then $\dfrac{dy}{dx}|_{x=\frac{\sqrt{\pi}}{2}} =$ $\sqrt{2\pi}$.

(d) If $y = \ln(|\sec x|)$ then $\dfrac{dy}{dx} =$ $\tan x$.

$\frac{1}{\sec x}(\sec x \tan x)$

(e) If $y = 2^{\sin x}$ then $\dfrac{dy}{dx} =$ $2^{\sin x} \cos x \ln 2$.

(f) If $g(x) = \dfrac{1}{x-2}$, $x \neq 2$ and $(f \circ g)(x) = x$ then $f(x) =$

$\frac{1}{x} + 2$.

(g) If $f(x) = |1 - x^2|$ then $f'(2) =$ 4 .

(h) If $f(x) = \dfrac{2x-3}{x-2}$, $x \neq 2$, the range of f is

all $y \neq 2$.

(i) The local linear approximation of $f(x) = \sqrt{9+x^2}$ at $x_0 = 0$ is

$y = 0$.

[4] 2. Find $\frac{dy}{dx}$ if $e^{xy} = x^2 + y^2$.

$e^{xy} = x^2 + y^2$

$\ln(e^{xy}) = \ln(x^2+y^2)$

$\therefore xy = \ln(x^2+y^2)$

Differentiate both sides to get

$y + xy' = \frac{1}{x^2+y^2}(2x+2yy')$

$\therefore y'\left(x - \frac{2y}{x^2+y^2}\right) = \frac{2x}{x^2+y^2} - y$

$y' = \dfrac{\frac{2x}{x^2+y^2} - y}{x - \frac{2y}{x^2+y^2}} = \dfrac{2x - y(x^2+y^2)}{x(x^2+y^2) - 2y}$

OR

$e^{xy} - x^2 - y^2 = 0$

$\frac{d}{dx}(e^{xy}) - 2x - 2y\,y' = 0$

$(y + xy')e^{xy} - 2x - 2yy' = 0$

$y'[xe^{xy} - 2y] = 2x - ye^{xy}$

$y' = \dfrac{2x - ye^{xy}}{xe^{xy} - 2y}$

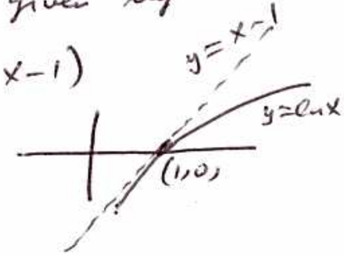
The 2 answers can be reconciled by using the equation of the curve, as in class

[3] 3. Find the equation of the tangent line to $y = \ln x$ at $x = 1$.

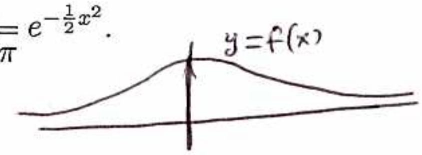
$y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$, $\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{x}\right|_{x=1} = 1$

When $x = 1$, $y = \ln 1 = 0$. So we want the equation of the tangent line at $\boxed{(1,0)}$. This is given by the formula $y - 0 = m(x-1)$ ie $y - 0 = 1(x-1)$

So $y = x - 1$



[3] 4. Draw a rough sketch of the graph of $y = f'(x)$ where $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.
[Hint: $f(x)$ is the usual "bell curve".]

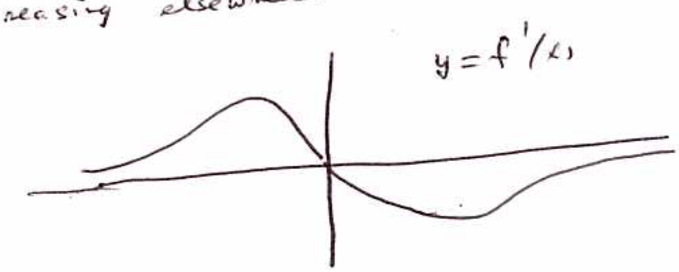


$f(x)$ is even so $f'(x)$ will be odd

$f''(x) = (x^2-1)e^{-1/2 x^2}$, so f'' is positive to the left of -1 and to the right of 1. ($x = \pm 1$ give inflections)

Thus, f' is increasing to the left of -1 and to the right of 1: f' is decreasing elsewhere.

Also $f'(0) = 0$



[2] 5. Evaluate the following limits.

(a) $\lim_{x\to 0}\left(x\sin\frac{1}{x}\right)$. $\underline{\quad 0 \quad}$

$|\sin\frac{1}{x}| \le 1$

(b) $\lim_{x\to 1}\left(\frac{\ln x}{x^2-5x+4}\right)$. $\underline{\quad -\frac{1}{3} \quad}$

$\frac{0}{0}$

[10] 6. True or False..

(a) For $\frac{\pi}{4} < x < \frac{\pi}{2}$, $2\cot x > 2$. $\underline{\quad F \quad}$.

$f(x) = 2\cot x - 2$, $f(\frac{\pi}{4}) = 0$, $f'(x) = -2\operatorname{cosec}^2 x < 0$, so f is decr. from 0

(b) If $\lim_{x\to a} f(x) = \infty$ then f cannot be continuous at $x = a$.

$\underline{\quad T \quad}$.

For continuity $f(a) = \lim_{x\to a} f(x)$ $(=\infty)$
But $f(a)$ must be a number, and ∞ is not a number

(c) $\log_4(e^x) = x\ln 4$. $\underline{\quad F \quad}$.

$\log_4(e^x) = x\log_e 4$

(d) There is no tangent to the curve $x^2 + y^2 = 1$ that passes through the point $(0,0)$. $\underline{\quad T \quad}$.

$(0,0)$ is the centre of the circle $x^2+y^2=1$.

(e) If x_1, x_2 are any non-zero numbers and $x_1 < x_2$ then it follows that $\frac{1}{x_1} < \frac{1}{x_2}$

$\underline{\quad F \quad}$.

Take examples or draw the graph of $y = \frac{1}{x}$

(f) The graph of $y = f(x+2)$ is obtained from the graph of $y = f(x)$ by translating 2 units to the left. $\underline{\quad T \quad}$.

(g) Suppose $f(x)$ is a cubic, i.e. $f(x) = ax^3 + bx^2 + cx + d$. Then it is not possible to find values $x_1 < x_2 < x_3 < x_4 < x_5$ such that $f(x_1)$, $f(x_3)$ and $f(x_5)$ are negative and such that $f(x_2)$ and $f(x_4)$ are positive.

$\underline{\quad T \quad}$.

if this were possible the cubic would have 4 roots.

(h) If $f(x) = \begin{cases} kx^2 + 2, & x \le 1 \\ 2kx - 1, & x > 1 \end{cases}$, then there is no value of k for which $f(x)$ is differentiable everywhere. $\underline{\quad F \quad (k=3)}$.

The left derivative will equal the right derivative no matter what value k has. However, for differentiability, we first need continuity so
$kx^2+2\big|_{x=1} = 2kx-1\big|_{x=1}$ giving $k=3$