Math. 249, Practice Mid Term March 1, 2005

(SOLNS)

1 Calculate the local linear approximation of

(18) $f(x) = \sqrt[3]{x+3}$ for values of x close to 5.

We have

$$f(x) \doteq f(a) + f'(a)(x-a)$$

Here, $f(x) = \sqrt[3]{x+3} = (x+3)^{\frac{1}{3}}$, $a = 5$

$$\therefore \sqrt[3]{x+3} \doteq (5+3)^{1/3} + \left[\frac{d}{dx}\left[(x+3)^{\frac{1}{3}}\right]\right]_{x=5} (x-5)$$

$$= 8^{1/3} + \overset{(*)}{\frac{1}{3}(x+3)^{-\frac{2}{3}}}\Big|_{x=5} (x-5)$$

$$= \sqrt[3]{8} + \frac{1}{3} \frac{1}{(\sqrt[3]{8})^2} (x-5)$$

$$= 2 + \frac{1}{3} \frac{1}{4} (x-5)$$

$$= 2 + \frac{1}{12}(x-5)$$

This gives the local linear approximation of $f(x) = \sqrt[3]{x+3}$ for values of x close to 5

(* $\frac{1}{3}(x+3)^{-\frac{2}{3}} = \frac{1}{3} \frac{1}{(x+3)^{\frac{2}{3}}} = \frac{1}{3} \frac{1}{\left[\sqrt[3]{x+3}\right]^2}$

since $(x+3)^{2/3} = \left[(x+3)^{1/3}\right]^2$

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2 Show that the equation $x^3 = x^2 - x + 1$ has at least one solution in the interval $[0,2]$, justifying your work.

(12)

Set $f(x) = x^3 - (x^2 - x + 1) = x^3 - x^2 + x - 1$

Now $f(x)$, being the difference of two continuous functions, is continuous.

We need to show that $x^3 = x^2 - x + 1$ has at least one soln. in $[0,2]$.

This is equivalent to showing that $f(x) = 0$ for at least one value of x in $[0,2]$

We have $f(0) = -1 < 0$

$f(2) = 2^3 - 2^2 + 2 - 1 = 10 - 5 = 5 > 0$

So f changes from being < 0 to being > 0.

By the IMT there is at least one x in $[0,2]$ with $f(x) = 0$ since f is continuous on $[0,2]$.

3

(a) Find $f'(x)$ if $f(x) = \dfrac{x^2+1}{x^3-x+7}$

2

$$f'(x) = \frac{\left[\frac{d}{dx}(x^2+1)\right](x^3-x+7) - (x^2+1)\frac{d}{dx}(x^3-x+7)}{(x^3-x+7)^2}$$

$$= \frac{(2x)(x^3-x+7) - (x^2+1)(3x^2-1)}{(x^3-x+7)^2}$$

(No need to simplify further)

(b) Find $f'(x)$ if $f(x) = \sec\left(\sqrt{x^2-x}\right)$

$$f'(x) = \sec\left(\sqrt{x^2-x}\right)\tan\left(\sqrt{x^2-x}\right)\frac{d}{dx}\left(\sqrt{x^2-x}\right)$$

$$= \sec\sqrt{x^2-x}\ \tan\sqrt{x^2-x}\ \frac{1}{2\sqrt{x^2-x}}\frac{d}{dx}(x^2-x)$$

$$= \sec\sqrt{x^2-x}\ \tan\sqrt{x^2-x}\ \frac{1}{2\sqrt{x^2-x}}(2x-1)$$

4 Let $f(x)$ be a differentiable function such that

12 $f(-1) = 2$, $f'(-1) = 6$. Find the equation of the tangent line to the curve $y = f(x)$ at $x = -1$

The equation of the tangent line at $(a, f(a))$

is $y - f(a) = \text{slope}\,(x - a)$

ie $y - f(a) = [f'(a)](x - a)$

Here $a = -1$, $f(a) = f(-1) = 2$, $f'(a) = f'(-1) = 6$

So we get

$$y - 2 = 6(x - (-1))$$

ie $y - 2 = 6(x + 1)$

PART B 35%

5 Find $\frac{dy}{dx}$ if $y^2 + \sin y = x$

$y^2 + \sin y - x = 0$, $\frac{d}{dx}\left(y^2 + \sin y - x\right) = \frac{d}{dx}(0) = 0$

$\therefore \frac{d}{dx}(y^2) + \frac{d}{dx}(\sin y) - 1 = 0$

This gives $2y\frac{dy}{dx} + \cos y \frac{dy}{dx} - 1 = 0$, $\frac{dy}{dx}\left[2y + \cos y\right] = 1$

$\frac{dy}{dx} = \frac{1}{2y + \cos y}$. $\left\{ \text{Note } \frac{d}{dx}\left(f(y)\right) = f'(y)\frac{dy}{dx} \right\}$

6 If $f(x) = \frac{\sqrt{4x^2+3}}{x-1}$ then the horizontal asymptotes are [get $\lim_{x \to \pm\infty} f(x)$] $\underline{y = \pm 2}$

Here we use the fact that $\sqrt{x^2} = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

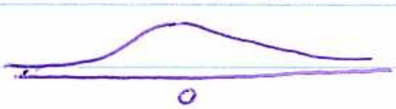
$\sqrt{4x^2} = \sqrt{4}\sqrt{x^2} = 2\sqrt{x^2}$

7 $\lim_{x \to 0}\left(\frac{|2x+4| - 4}{x}\right) = \underline{\quad 2 \quad}$

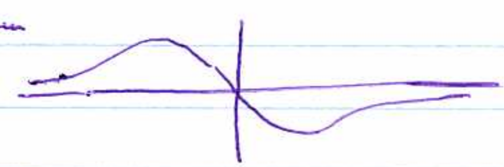
when x is close to 0, $2x+4$ is positive so we have $|2x+4| = 2x+4$ in that case; $\frac{2x+4-4}{x} = 2$

8 If $f(x)$ is the usual bell curve then a rough sketch of $f'(x)$ is as follows.

See soln. in class and on old 251 midterm

$f(x) =$ 

// even

$f'(x) =$ 

// odd

9 $\lim_{x \to \frac{\pi}{4}}\left(\frac{\tan x - \tan\frac{\pi}{4}}{x - \pi/4}\right) = \frac{d}{dx}(\tan x)\Big|_{x=\pi/4}$ $\underline{\quad 2 \quad}$

$= \sec^2\frac{\pi}{4}$

$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$

10 True or False: $f(x) = \sin x$ is continuous for all x

(unlike e.g. $\tan x = \frac{\sin x}{\cos x}$ which is discontinuous when $\cos x = 0$) $\underline{\quad \text{True} \quad}$