MATH 251

PRACTICE MIDTERM - SOLUTIONS

1. Solve the following inequality:

$$\text{(a) } |x-2|<1, \qquad \text{(b) } x-2 \le \frac{-1}{x+1}.$$

Solution: (a) The distance between x and 2 is less than 1, so $x \in (2-1, 2+1) = (1,3)$.

OR: If the absolute value of a number is less than 1, it must lie between -1 and 1, so $-1 < x-2 < 1$. Now $-1+2 < x < 1+2$, $1 < x < 3$.

Answer: $x \in (1,3)$.

(b) First bring everything to one side:

$$x - 2 + \frac{1}{x+1} \le 0.$$

Simplify:

$$\frac{(x-2)(x+1)+1}{x+1} \le 0; \qquad \frac{x^2-x-1}{x+1} \le 0; \qquad \frac{(x-\frac{1+\sqrt{5}}{2})(x-\frac{1-\sqrt{5}}{2})}{x+1} \le 0.$$

Testing:

$$\underline{\quad - \quad}\ -\hat{1}\ \underline{\quad + \quad}\ \tfrac{1-\sqrt{5}}{2}\ \underline{\quad - \quad}\ \tfrac{1+\sqrt{5}}{2}\ \underline{\quad + \quad}$$

Here $-1 < \frac{1-\sqrt{5}}{2}$, since $-2 < 1-\sqrt{5}$, since $\sqrt{5} < 3$, since $5 < 9$.

The point $x=-1$ does not belong to the solution set, since it is not in the domain.

Answer: $x \in (-\infty, -1) \cup [\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}]$.

2. Find all horizontal and vertical asymptotes of

$$\text{(a) } f(x) = \frac{x^2}{x-1} - \frac{x^2+1}{x+1}, \qquad \text{(b) } f(x) = \frac{\sqrt{9x^2+x+5}}{x+2}.$$

Solution: A vertical asymptote of the graph of $y=f(x)$ is such a vertical line $x=a$ that either $\lim_{x\to a^-} f(x) = \infty$, or $\lim_{x\to a^-} f(x) = -\infty$, or $\lim_{x\to a^+} f(x) = \infty$, or $\lim_{x\to a^+} f(x) = -\infty$. In particular, if $x=a$ is a vertical asymptote of f then f must be discontinuous at $x=a$.

A horizontal asymptote of the graph of $y=f(x)$ is such a horizontal line $y=b$ that either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$. In particular, there can be at most two horizontal asymptotes, namely the horizontal asymptotes $y=b_1$ and $y=b_2$ if $\lim_{x\to\infty} f(x) = b_1$, $\lim_{x\to-\infty} f(x) = b_2$, and $b_1 \ne b_2$ (if $b_1 = b_2$ then $y=b_1$ is a unique horizontal asymptote).

(a) Since f is continuous at all points $x \neq 1, -1$, the only vertical asymptotes of the graph of $y = f(x)$ can be $x = 1$ or $x = -1$. We still must check whether $x = 1$ and $x = -1$ are indeed vertical asymptotes, so we must find out whether a one-sided limit of f at these points is ∞ or $-\infty$ or neither. We have

$$\lim_{x\to 1^+} = \text{“}\frac{1}{0^+} - \frac{2}{2}\text{”} = \text{“}\infty - 1\text{”} = \infty$$

(and $\lim_{x\to 1^-} = \text{“}\frac{1}{0^-} - \frac{2}{2}\text{”} = \text{“} - \infty - 1\text{”} = -\infty$), so the line $x = 1$ is indeed a vertical asymptote;

$$\lim_{x\to -1^+} = \text{“}\frac{1}{-1-1} - \frac{2}{0^+}\text{”} = \text{“} - \frac{1}{2} - \infty\text{”} = -\infty$$

(and $\lim_{x\to -1^-} = \text{“}\frac{1}{-1-1} - \frac{2}{0^-}\text{”} = \text{“} - \frac{1}{2} + \infty\text{”} = \infty$), so the line $x = -1$ is indeed a vertical asymptote.

To find the horizontal asymptotes, let us calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x-1} - \frac{x^2+1}{x+1} = \text{“}\frac{\infty}{\infty} - \frac{\infty}{\infty}\text{”}.$$

Since the limit has indefinite type, we must simplify f:

$$f(x) = \frac{x^2}{x-1} - \frac{x^2+1}{x+1} = \frac{x^3 + x^2 - x^3 + x^2 - x + 1}{(x-1)(x+1)} = \frac{2x^2 - x + 1}{x^2 - 1}$$

Thus

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2 - x + 1}{x^2 - 1} = \text{“}\frac{\infty}{\infty}\text{”}$$

Divide both numerator and denominator by x^2, which is the highest power of x:

$$= \lim_{x\to\infty} \frac{2 - \frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{2 - 0 + 0}{1 - 0} = 2.$$

Similarly

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2 - \frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{2 - 0 + 0}{1 - 0} = 2.$$

Since both limits of $f(x)$ as $x \to \infty$ and $x \to -\infty$ equal 2, the graph of $y = f(x)$ has a unique horizontal asymptote $y = 2$.

Answer: The vertical asymptotes are $x = -1$ and $x = 1$, and the horizontal asymptote is $y = 2$.

(b) First find the domain of f. The discriminant of $9x^2 + x + 5$ equals $1 - 4 \cdot 9 \cdot 5 < 0$, so $9x^2 + x + 5$ is always positive, thus the numerator is continuous for all $x \in \mathbb{R}$. So the domain of f is $D_f = (-\infty, -2) \cup (-2, \infty)$.

Since f is continuous in its domain, the only vertical asymptote of the graph of $y = f(x)$ can be $x = -2$. We still must check whether $x = -2$ is indeed a vertical asymptote, so we must find out whether a one-sided limit of f at this point is ∞ or $-\infty$ or neither:

$$\lim_{x\to -2^+} = \text{“}\frac{\sqrt{36 - 2 + 5}}{0^+}\text{”} = \text{“}\frac{\sqrt{39}}{0^+}\text{”} = \infty$$

(and $\lim_{x\to -2^-} = $ “$\frac{\sqrt{36-2+5}}{0^-}$” $=$ “$\frac{\sqrt{39}}{0^-}$” $= -\infty$), so the line $x = -2$ is indeed a vertical asymptote.

To find the horizontal asymptotes, let us calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{9x^2+x+5}}{x+2} = \text{“}\frac{\infty}{\infty}\text{”}$$

If x is very large, $9x^2+x+5$ behaves as $9x^2$, so the numerator behaves as $\sqrt{9x^2} = 3x$ as $x \to \infty$. Divide both numerator and denominator by x, which is the highest power of x:

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{9x^2+x+5}}{x}}{1+\frac{2}{x}}$$

Since $x \to \infty$, x is positive, so we have $x = \sqrt{x^2}$ (with the plus sign), so

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{9x^2+x+5}}{\sqrt{x^2}}}{1+\frac{2}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{9x^2+x+5}{x^2}}}{1+\frac{2}{x}} = \lim_{x\to\infty} \frac{\sqrt{9+\frac{1}{x}+\frac{5}{x^2}}}{1+\frac{2}{x}} = \frac{\sqrt{9+0+0}}{1+0} = 3.$$

Similary

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{9x^2+x+5}}{x+2} = \text{“}\frac{\infty}{-\infty}\text{”} = \lim_{x\to-\infty} \frac{\frac{\sqrt{9x^2+x+5}}{x}}{1+\frac{2}{x}}$$

Since $x \to -\infty$, x is negative, so we have $x = -\sqrt{x^2}$ (with the minus sign), so

$$= \lim_{x\to-\infty} \frac{\frac{\sqrt{9x^2+x+5}}{-\sqrt{x^2}}}{1+\frac{2}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{9x^2+x+5}{x^2}}}{1+\frac{2}{x}} = \lim_{x\to-\infty} -\frac{\sqrt{9+\frac{1}{x}+\frac{5}{x^2}}}{1+\frac{2}{x}} = -\frac{\sqrt{9+0+0}}{1+0} = -3.$$

Thus the horizontal asymptotes of the graph of $y = f(x)$ are $y = 3$ and $y = -3$.

Answer: The vertical asymptote is $x = -2$, and the horizontal asymptotes are $y = 3$ and $y = -3$.

3. Find the following limits:

(a)
$$\lim_{x\to-\infty} \frac{x^8 - 3x^5 + 2}{2x^8 + x^5 - 3} = \lim_{x\to-\infty} \frac{1 - \frac{3}{x^3} + \frac{2}{x^8}}{2 + \frac{1}{x^3} - \frac{3}{x^8}} = \frac{1-0+0}{2+0-0} = \frac{1}{2}.$$

(b)
$$\lim_{x\to 2} \frac{x^3 - x^2 - 4}{x^2 - 4} = \text{“}\frac{8-4-4}{4-4}\text{”} = \text{“}\frac{0}{0}\text{”}$$

Since $x = 2$ is a root of the numerator, $x - 2$ is a factor of the numerator. To find another factor, we use long division of polynomials, thus $x^3 - x^2 - 4 = (x-2)(x^2+x+2)$ and

$$= \lim_{x\to 2} \frac{(x-2)(x^2+x+2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x^2+x+2}{x+2} = \frac{4+2+2}{2+2} = 2.$$

(c) $$\lim_{x\to-2^-}\frac{2x^2+3x+1}{x^3+x^2+4} = \text{“}\frac{8-6+1}{-8+4+4}\text{”} = \text{“}\frac{3}{0}\text{”}$$

We see that the limit DOES NOT EXIST, but we still must find out whether it is ∞ or $-\infty$ or neither. Since $x = -2$ is a root of the denominator, $x+2$ is a factor of the denominator. To find another factor, we use long division of polynomials, thus $x^3+x^2+4 = (x+2)(x^2-x+2)$ and

$$= \lim_{x\to-2}\frac{2x^2+3x+1}{(x+2)(x^2-x+2)} = \frac{8-6+1}{(0^-)(4+2+2)} = \frac{3}{8(0^-)} = -\infty, \qquad \textbf{DNE}.$$

(d)
$$\lim_{x\to0}\frac{|x-3|-3}{2x} = \lim_{x\to0}\frac{-(x-3)-3}{2x} = \lim_{x\to0}\frac{-x+3-3}{2x} = \lim_{x\to0}\frac{-x}{2x} = \lim_{x\to0}\frac{-1}{2} = -\frac{1}{2}.$$

(e) $$\lim_{x\to\infty}\frac{3x-\cos x}{x} = \lim_{x\to\infty}(3-\frac{\cos x}{x})$$

Let us find the limit of $\frac{\cos x}{x}$ as $x\to\infty$. Since $-\frac{1}{x}\le\frac{\cos x}{x}\le\frac{1}{x}$ for $x>0$, and the “squeezing” functions $-\frac{1}{x}$ and $\frac{1}{x}$ have the same limit 0 as $x\to\infty$, the function $\frac{\cos x}{x}$ is forced to have the same limit **by the Squeeze Theorem**. So, $\lim_{x\to\infty}\frac{\cos x}{x} = 0$ and

$$= 3-0 = 3.$$

(f) $$\lim_{x\to\frac{\pi}{4}}\frac{\tan x-\tan\frac{\pi}{4}}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}}\text{“}\frac{0}{0}\text{”}$$

The function $\frac{\tan x-\tan\frac{\pi}{4}}{x-\frac{\pi}{4}}$ is the Newton quotient $\frac{f(x)-f(a)}{x-a}$ for the function $f(x)=\tan x$ and the point $a=\frac{\pi}{4}$. Since the Newton quotient $\frac{f(x)-f(a)}{x-a}$ approaches $f'(a)$ as $x\to a$, we have

$$= f'(\frac{\pi}{4}) = (\tan x)'|_{x=\frac{\pi}{4}} = \sec^2\frac{\pi}{4} = \frac{1}{\cos^2\frac{\pi}{4}} = \frac{1}{1/2} = 2.$$

(g) $$\lim_{x\to\infty}x\sin\frac{1}{x} = \lim_{t\to0}\frac{1}{t}\sin t = \lim_{t\to0}\frac{\sin t}{t} = 1.$$

(h) $$\lim_{x\to0}x\sin\frac{1}{x}$$

Since $-|x|\le x\sin\frac{1}{x}\le|x|$ for any $x\neq0$, and the “squeezing” functions $-|x|$ and $|x|$ have the same limit 0 as $x\to0$, the function $x\sin\frac{1}{x}$ is forced to have the same limit **by the Squeeze Theorem**, so

$$= 0.$$

(i) $$\lim_{x\to 0}\sin\frac{1}{x} \qquad \text{DNE},$$

since the function $y=\sin\frac{1}{x}$ oscillates between $y=-1$ and $y=1$ infinitely many times as $x\to 0$.

4. Determine $k\in\mathbb{R}$ such that the function $\quad g(x)=\begin{cases} x^2-4k & \text{if } x\le 1,\\ x^3+k^2+4 & \text{if } x>1\end{cases}$
is continuous.

Solution: The function g is piece-wise defined. Since g is a polynomial in the interval $(-\infty,1]$, it is continuous at any point $x<1$. Since g is a polynomial in the interval $(1,\infty)$, it is continuous at any point $x>1$. The continuity at $x=1$ means that $\lim_{x\to 1^-}g(x)=\lim_{x\to 1^+}g(x)=g(1)$. Now

$$\lim_{x\to 1^-}g(x)=g(1)=(x^2-4k)|_{x=1}=1-4k,$$

$$\lim_{x\to 1^+}g(x)=(x^3+k^2+4)|_{x=1}=1+k^2+4=k^2+5.$$

So we must solve the equation $\quad 1-4k=k^2+5 \quad$ for k. We have $k^2+4k+4=0$, $(k+2)^2=0$, $k=-2$.

Answer: $k=-2$.

5. Differentiate the following functions. If possible, simplify your answers.

(a) $$y=3x^4-10x^{6/5},\qquad y'=3\cdot 4x^3-10\cdot\frac{6}{5}x^{1/5}=12x^3-12x^{1/5}.$$

(b) $$y=\frac{-2}{x^3+2},\qquad y'=\frac{2}{(x^3+2)^2}\cdot 3x^2=\frac{3x^2}{(x^3+2)^2}.$$

(c) $$y=\frac{x^2+1}{x^3-x+5},\qquad y'=\frac{2x(x^3-x+5)-(3x^2-1)(x^2+1)}{(x^3-x+5)^2}=\frac{-x^4-4x^2+10+1}{(x^3-x+5)^2}.$$

(d) $$y=|2x|=\begin{cases}2x & \text{if } x>0\\ 0 & \text{if } x=0\\ -2x & \text{if } x<0\end{cases},\qquad y'=\begin{cases}2 & \text{if } x>0\\ \text{DNE} & \text{if } x=0\\ -2 & \text{if } x<0\end{cases}$$

Here $y'(0)=\lim_{x\to 0}\frac{|2x|}{x}$, but $\lim_{x\to 0^+}\frac{|2x|}{x}=\frac{2x}{x}=2$ and $\lim_{x\to 0^-}\frac{|2x|}{x}=\frac{-2x}{x}=-2$, thus the limit of $\frac{|2x|}{x}$ as $x\to 0$ does not exist, thus $y'(0)$ DOES NOT EXIST.

(e) $$y=x^2\sin(3x)+\cot^2(\frac{1}{x}),\qquad y'=2x\sin(3x)+x^2\cos(3x)\cdot 3+2\cot(\frac{1}{x})\cdot(-\csc^2(\frac{1}{x}))\cdot(-\frac{1}{x^2})$$

$$=2x\sin(3x)+3x^2\cos(3x)+\frac{2}{x^2}\cot(\frac{1}{x})\cdot\csc^2(\frac{1}{x}).$$

(f) $y = \dfrac{\tan(3x)}{x+7}, \qquad y' = \dfrac{\sec^2(3x)\cdot 3(x+7) - \tan(3x)}{(x+7)^2} = \dfrac{3(x+7)\sec^2(3x) - \tan(3x)}{(x+7)^2}.$

(g) $y = \dfrac{\cos x + 1}{\sin x + 1}, \quad y' = \dfrac{-\sin x(\sin x + 1) - \cos x(\cos x + 1)}{(\sin x + 1)^2} = \dfrac{-\sin^2 x - \sin x - \cos^2 x - \cos x}{(\sin x + 1)^2}$

$$= \frac{-(\sin^2 x + \cos^2 x) - \sin x - \cos x}{(\sin x + 1)^2} = \frac{-1 - \sin x - \cos x}{(\sin x + 1)^2} = -\frac{1 + \sin x + \cos x}{(\sin x + 1)^2}.$$

(h) $y = \cos^3(x)\cos(x^3), \qquad y' = 3\cos^2 x \cdot (-\sin x)\cos(x^3) + \cos^3 x \cdot (-\sin(x^3)\cdot 3x^2)$

$$= -3\cos^2 x \cdot \sin x \cdot \cos(x^3) - 3x^2\cos^3 x \cdot \sin(x^3).$$

(i) $y = \sec^2(x^2 + 5x), \qquad y' = 2\sec(x^2 + 5x)\cdot\sec(x^2 + 5x)\cdot\tan(x^2 + 5x)\cdot(2x + 5)$

$$= (4x + 10)\sec^2(x^2 + 5x)\cdot\tan(x^2 + 5x).$$

(j) $y = \csc(\sqrt{x^2 + x}), \qquad y' = -\csc(\sqrt{x^2 + x})\cdot\cot(\sqrt{x^2 + x})\cdot\dfrac{2x + 1}{2\sqrt{x^2 + x}}$

$$= -\frac{2x + 1}{2\sqrt{x^2 + x}}\ \csc(\sqrt{x^2 + x})\cdot\cot(\sqrt{x^2 + x}).$$

(k) $y = \sin^3(\cos x), \qquad y' = 3\sin^2(\cos x)\cdot\cos(\cos x)\cdot(-\sin x) = -3(\sin x)\sin^2(\cos x)\cdot\cos(\cos x).$

6. (a) Estimate $\sqrt{103}$ using differentials.

Solution: Since $\sqrt{100} = 10$, it is useful to approximate the function $y = f(x) = \sqrt{x}$ near the point $x = a = 100$.

METHOD 1: Let us approximate the graph of $y = f(x)$ by its tangent line at the point $(a, f(a)) = (100, 10)$. The equation of this tangent line is

$$y = L(x) \qquad \text{where} \qquad L(x) = f(a) + f'(a)\cdot(x - a).$$

The function $L(x)$ is called the *linear approximation* of $f(x)$ at the point $x = a$. Now

$$L(x) = \sqrt{100} + \frac{x - 100}{2\sqrt{100}} = 10 + \frac{x - 100}{20}.$$

Since $f(x) \approx L(x)$ if the point x is close enough to a, we have

$$\sqrt{103} = f(103) \approx L(103) = 10 + \frac{3}{20} = 10.15$$

METHOD 2: If x increases at $\Delta x = dx$ then y increases at $\Delta y = f(x + \Delta x) - f(x)$. Since the Newton quotient $\frac{\Delta y}{\Delta x}$ approaches $f'(x)$ as Δx approaches 0, we have

$$f'(x) \approx \frac{\Delta y}{\Delta x}, \qquad \Delta y \approx f'(x)\cdot\Delta x.$$

So $\Delta y \approx \frac{dy}{dx}dx = dy$. Let us find dy for $x = 100$ and $dx = 3$. Since $f(x) = \sqrt{x}$, we have

$f'(x) = \frac{1}{2\sqrt{x}}$, thus

$$dy|_{x=100,\ dx=3} = f'(x) \cdot dx|_{x=100,\ dx=3} = \left.\frac{dx}{2\sqrt{x}}\right|_{x=100,\ dx=3} = \frac{3}{20} = 0.15$$

Since $y = f(100) = \sqrt{100} = 10$ and $\Delta y \approx dy$, we have

$$\sqrt{103} = (y + \Delta y)|_{x=100,\ dx=3} \approx (y + dy)|_{x=100,\ dx=3} = 10 + 0.15 = 10.15$$

Answer: $\sqrt{103} \approx 10.15$

(b) Find the linear approximation to $f(x) = -x^2 + \cos x$ at $x_0 = \pi$. Find the Error of this linear approximation at $x = -\pi$.

Solution: Let us approximate the graph of $y = f(x)$ by its tangent line at the point $(x_0, y_0) = (\pi, f(\pi))$. The equation of this tangent line is

$$y = L(x) \qquad \text{where} \qquad L(x) = f(\pi) + f'(\pi) \cdot (x - \pi).$$

The function $L(x)$ is called the *linear approximation* of $f(x)$ at the point $x = \pi$. Since $f(\pi) = -\pi^2 - 1$ and $f'(x) = -2x - \sin x$, we have $f'(\pi) = -2\pi$ and

$$L(x) = -\pi^2 - 1 - 2\pi(x - \pi) = -\pi^2 - 1 - 2\pi x + 2\pi^2 = \pi^2 - 1 - 2\pi x$$

Since $f(x) \approx L(x)$ for any point x which is close enough to π, we have

$$f(x) \approx \pi^2 - 1 - 2\pi x \qquad \text{if } x \text{ is close enough to } \pi.$$

The Error of this linear approximation at the point $x = -\pi$ equals

$$|f(-\pi) - L(-\pi)| = |-\pi^2 - 1 - (\pi^2 - 1 + 2\pi^2)| = |-4\pi^2| = 4\pi^2 \approx 36.$$

We got a large Error, since $-\pi$ is far away from π.

7. Find all intervals where the function $f(x) = x^3 - 3x^2 - 6x + 2$ is increasing or decreasing.

Solution: $f'(x) = 3x^2 - 6x - 6 = 3(x^2 - 2x - 2)$. The roots of $f'(x)$ are $x = 1 + \sqrt{1+2} = 1 + \sqrt{3}$ and $x = 1 - \sqrt{3}$. Testing:

f'	$+$	$1-\sqrt{3}$	$-$	$1+\sqrt{3}$	$+$

So

f	increases	decreases	increases

Answer: f is increasing on $(-\infty, 1 - \sqrt{3}]$ and $[1 + \sqrt{3}, \infty)$, and decreasing on $[1 - \sqrt{3}, 1 + \sqrt{3}]$.

8. Given the data in the following table, determine $(g \circ f)'(2)$

x	-1	0	2	5
$f(x)$	2	1	-1	7
$g(x)$	5	2	0	4
$f'(x)$	0	2	3	-1
$g'(x)$	4	5	2	5

Solution: By the Chain Rule, $(g \circ f)'(x) = (g(f(x)))' = g'(f(x)) \cdot f'(x)$. So,

$$(g \circ f)'(2) = g'(f(2)) \cdot f'(2)$$

From the table, we get $f(2) = -1$ and $f'(2) = 3$, so

$$= g'(-1) \cdot 3$$

From the table, we get $g'(-1) = 4$, so

$$= 4 \cdot 3 = 12.$$

Answer: 12.

9. Find the equation of the tangent line to the graph of

(a) $\tan(xy) + \sin y = x^2 y + 2x - 2$ at the point $(1, 0)$;

(b) $y^2 + \sin y = x$ at the point (π^2, π).

Solution: (a) First check that the point $(1, 0)$ satisfies the given equation: for $x = 1, y = 0$ the left hand side equals $\tan(0) + \sin 0 = 0 + 0 = 0$ and the right hand side equals $0 + 2 - 2 = 0$, so the point $(1, 0)$ belongs to the graph. Let us differentiate both sides of $\tan(xy) + \sin y = x^2 y + 2x - 2$ with respect to x using the chain rule, remembering that y is a function of x:

$$\sec^2(xy) \cdot (y + xy') + \cos y \cdot y' = 2xy + x^2 y' + 2.$$

In order to solve this equation for y', we bring all terms containing y' to the left, and bring all terms which do not contain y' to the right:

$$(x \sec^2(xy) + \cos y - x^2) y' = -y \sec^2(xy) + 2xy + 2, \qquad y' = \frac{-y \sec^2(xy) + 2xy + 2}{x \sec^2(xy) + \cos y - x^2}.$$

Substituting $x = 1, y = 0$, we obtain the slope of the tangent line at the point $(1, 0)$:

$$m = y'(1) = \left. \frac{-y \sec^2(xy) + 2xy + 2}{x \sec^2(xy) + \cos y - x^2} \right|_{\substack{x=1 \\ y=0}} = \frac{\sec 0 + 0 + 2}{\sec 0 + \cos 0 - 1} = \frac{1 + 0 + 2}{1 + 1 - 1} = 3.$$

So, the tangent line at the point $(1,0)$ has slope $m = 3$. Therefore the equation of this tangent line is $y = m(x-1) = 3(x-1) = 3x-3$.

Answer: $y = 3x - 3$.

(b) First check that the point (π^2, π) satisfies the given relation: $\pi^2 + \sin\pi = \pi^2$, so the point (π^2, π) belongs to the graph. Let us differentiate both sides of $y^2 + \sin y = x$ with respect to x using the chain rule, remembering that y is a function of x:

$$2yy' + y'\cos y = 1, \qquad y'(2y + \cos y) = 1, \qquad y' = \frac{1}{2y + \cos y}.$$

Substituting $x = \pi^2, y = \pi$, we obtain the slope of the tangent line at the point (π^2, π):

$$m = y'(\pi^2) = \frac{1}{2y + \cos y}\Bigg|_{\substack{x=\pi^2\\ y=\pi}} = \frac{1}{2\pi - 1}.$$

So, the tangent line at the point (π^2, π) has slope $m = \frac{1}{2\pi-1}$. Therefore the equation of this tangent line is $y = m(x-\pi^2) + \pi = \frac{x-\pi^2}{2\pi-1} + \pi = \frac{x-\pi^2+2\pi^2-\pi}{2\pi-1} = \frac{x+\pi^2-\pi}{2\pi-1}$.

Answer: $y = \frac{x+\pi^2-\pi}{2\pi-1}$.

10. Answer TRUE or FALSE for questions (a)–(e) and (g)–(m). **Do not write "T, F".**

(a) If x_1, x_2 are any non-zero numbers and $x_1 < x_2$ then it follows that $\frac{1}{x_1} > \frac{1}{x_2}$. FALSE

(b) $2.\overline{9} < 3$ FALSE

(c) If f, g are discontinuous at $x = a$ then $f + g$ is discontinuous at $x = a$. FALSE

(d) If $\lim_{x\to a} f(x) = \infty$ then f can not be continuous at $x = a$. TRUE

(e) If $g(x) = (f(x))^2$ is continuous then $f(x)$ must also be continuous. FALSE

(f) The Intermediate-Value Theorem allows us to conclude that the equation $3x^3 - x + 4 = 5$ has a solution in the interval
a: $[-1, 0]$, b: $[0, 1]$, c: $[1, 2]$, d: None of these. b

(g) If f is continuous then f is differentiable. FALSE

(h) If $f(x) > g(x)$ then $f'(x) > g'(x)$. FALSE

(i) $(fgh)' = f'gh + fg'h + fgh'$. TRUE

(j) The function $f(x) = \sin x$ is continuous for all x. TRUE

(k) If $\frac{\pi}{4} < x < \frac{\pi}{2}$ then $\tan x > 1$. TRUE

(l) $\tan(\frac{21\pi}{4}) > 0$ TRUE

(m) $\tan(3.24) > 0$ TRUE

(n) If $f(x) = \frac{2x-3}{x-2}$ for $x \neq 2$ then the range of f is $(-\infty, 2) \cup (2, \infty)$

(o) If $f(x) = \frac{1}{x+2}$ for $x \neq -2$ and $(f \circ g)(x) = x$ then $g(x) =$ $\frac{1}{x} - 2$