MATH 251 **L12, L14** **Winter 2005**

MIDTERM **March 15, 2005** **70 minutes**

Name: K E Y (Please write your ID# on p. 2)

Answer all questions. Show ALL relevant work, and circle your answers. No calculators or other aids. Marks are shown in brackets.

1. Differentiate the following functions. If possible, simplify your answers. [12]

(a) $$y = x^3\sqrt{x+2}$$

Solution:

$$y' = 3x^2\sqrt{x+2} + x^3 \cdot \frac{1}{2\sqrt{x+2}} = \frac{6x^2(x+2)+x^3}{2\sqrt{x+2}} = \frac{7x^3+12x^2}{2\sqrt{x+2}} = x^2\frac{7x+12}{2\sqrt{x+2}}$$

(b) $$y = \sin^3(\cos x)$$

Solution:

$$y' = 3\sin^2(\cos x)\cdot\cos(\cos x)\cdot(-\sin x) = -3\sin^2(\cos x)\cdot\cos(\cos x)\cdot\sin x$$

2. Determine $k \in \mathbb{R}$ such that the function $g(x) = \begin{cases} kx^2 - x & \text{if } x < 2, \\ x^3 + k^2 - 6 & \text{if } x \geq 2 \end{cases}$ [9]

is continuous.

Solution: The function g is piece-wise defined. Since g is a polynomial in the interval $(-\infty, 2)$, it is continuous at any point $x < 2$. Since g is a polynomial in the interval $[2, \infty)$, it is continuous at any point $x > 2$. The continuity at $x = 2$ means that $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = g(2)$. Now

$$\lim_{x\to 2^-} g(x) = (kx^2 - x)|_{x=2} = 4k - 2,$$

$$\lim_{x\to 2^+} g(x) = g(2) = (x^3 + k^2 - 6)|_{x=2} = 8 + k^2 - 6 = k^2 + 2.$$

So we must solve the equation $4k - 2 = k^2 + 2$ for k. We have $k^2 - 4k + 4 = 0$, $(k-2)^2 = 0$, $k = 2$.

Answer: $k = 2$.

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Explanation for question 3:

(a) Since x_1 and x_2 are positive, $x_1 x_2$ is also positive. So we may divide both sides of $x_1 < x_2$ by $x_1 x_2$, and the inequality will not change sign:
$x_1 < x_2$, $\frac{x_1}{x_1 x_2} < \frac{x_2}{x_1 x_2}$, $\frac{1}{x_2} < \frac{1}{x_1}$, therefore $\frac{1}{x_1} > \frac{1}{x_2}$.

(b) The number π is irrational, so its decimal expansion is neither terminating nor periodic. Actually $\pi = 3.14158265358\ldots$

(c) Let $f(x) = \operatorname{sgn}(x)$ for $x \neq 0$, and $f(0) = 1$. Then $f(x)$ is discontinuous at $x = 0$, but $(f(x))^2 = 1$ for any x, so $(f(x))^2$ is continuous.

Actually, the converse of the given statement is true: if $f(x)$ is continuous then $g(x) = (f(x))^2$ must also be continuous. (Since the product of continuous functions is always continuous.)

(d) Suppose that f is differentiable. Let us show that f is continuous at any point $x = a$ of its domain. We need to show that $\lim_{x\to a} f(x) = f(a)$. Let us find $\lim_{x\to a} f(x) - f(a)$:

$$\lim_{x\to a} f(x) - f(a) = \lim_{x\to a}(f(x) - f(a)) = \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right)$$

Since $f(x)$ is differentiable at $x = a$, the Newton quotient $\frac{f(x)-f(a)}{x-a}$ approaches $f'(a)$ as $x \to a$. Since $x - a$ approaches 0 as $x \to a$, we have

$$= \left(\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right) \cdot \lim_{x\to a}(x - a) = f'(a) \cdot 0 = 0.$$

(e) Take $f(x) = x$ and $g(x) = 0$. Then $f'(x) = 1 > 0 = g'(x)$. But $f(-1) < g(-1)$.

(f) The function $f(x) = |3x|$ is piece-wise defined: $f(x) = 3x$ for $x \geq 0$, and $f(x) = -3x$ for $x < 0$. So f is continuous at all points $x \neq 0$. Since $\lim_{x\to 0^-} f(x) = (-3x)|_{x=0} = 0$ and $\lim_{x\to 0^+} f(x) = f(0) = 0$, f is also continuous at $x = 0$.

(g) The correct rule is $(fgh)' = f'gh + fg'h + fgh'$.

(h) Since $\frac{23\pi}{4} = 6\pi - \frac{\pi}{4}$, we have $\tan(\frac{23\pi}{4}) = \tan(-\frac{\pi}{4}) = -1$.

(i) Since f is a polynomial, it is continuous. Let us test the values of $f(x)$ at the points $x = -1, 0, 1, 2$: we get $f(-1) = -2 + 1 - 2 = -3$, $f(0) = -2$, $f(1) = 2 - 1 - 2 = -1$, $f(2) = 16 - 2 - 2 = 12$.

So, the values of f have different signs at the endpoints of the interval $[1, 2]$. The Intermediate-Value Theorem allows us to conclude that the equation $f(x) = 0$ has a root in this interval.

(j) Since $f(x) = \frac{3x-2}{x-2} = \frac{3(x-2)+4}{x-2} = 3 + \frac{4}{x-2}$, the graph of f is a hyperbola with the vertical asymptote $x = 2$ and the horizontal asymptote $y = 3$. So, any number $y \neq 3$ is an output value of $y = f(x)$.

3. Answer TRUE or FALSE for (a)–(h). **DO NOT write "T,F".**
For (i) just answer a, b, c, or d, and for (j) give the appropriate interval(s).
No explanation is necessary for this question. [20]

(a) If x_1, x_2 are any positive numbers and $x_1 < x_2$ then it follows that $\frac{1}{x_1} > \frac{1}{x_2}$. TRUE

(b) $\pi = 3.14$ FALSE

(c) If $g(x) = (f(x))^2$ is continuous then $f(x)$ must also be continuous. FALSE

(d) If f is differentiable then f is continuous. TRUE

(e) If $f'(x) > g'(x)$ then $f(x) > g(x)$. FALSE

(f) The function $f(x) = |3x|$ is continuous for all x. TRUE

(g) $(fgh)' = f'g'h + f'gh' + fg'h'$. FALSE

(h) $\tan(\frac{23\pi}{4}) > 0$. FALSE

(i) The Intermediate-Value Theorem allows us to conclude that the equation $2x^3 - x - 2 = 0$ has a solution in the interval
a: $[-1, 0]$, b: $[0, 1]$, c: $[1, 2]$, d: None of these. c

(j) If $f(x) = \frac{3x-2}{x-2}$ for $x \neq 2$ then the range of f is $(-\infty, 3) \cup (3, \infty)$

4. Estimate $\sqrt[3]{997}$ using differentials. [10]

Solution: Since $\sqrt[3]{1000} = 10$, it is useful to approximate the function $y = f(x) = \sqrt[3]{x}$ near the point $x = a = 1000$. Let us approximate the graph of $y = f(x)$ by its tangent line at the point $(a, f(a)) = (1000, 10)$. The equation of this tangent line is

$$y = L(x) \qquad \text{where} \qquad L(x) = f(a) + f'(a) \cdot (x - a).$$

The function $L(x)$ is called the *linear approximation* of $f(x)$ at the point $x = a$. Now

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}},$$

so

$$L(x) = \sqrt[3]{1000} + \frac{x - 1000}{3\sqrt[3]{1000^2}} = \sqrt[3]{1000} + \frac{x - 1000}{3(\sqrt[3]{1000})^2} = 10 + \frac{x - 1000}{300}.$$

Since $f(x) \approx L(x)$ if the point x is close enough to a, we have

$$\sqrt[3]{997} = f(997) \approx L(997) = 10 + \frac{-3}{300} = 10 - \frac{1}{100} = 9.99$$

Answer: $\sqrt[3]{997} \approx 9.99$

5. Solve the following inequalities: [10]

(a)
$$|x-3|<2$$

Solution: We show three different ways to solve this simple inequality.

METHOD 1: The distance between x and 3 is less than 2, so $x \in (3-2, 3+2) = (1,5)$.

METHOD 2: If the absolute value of a number is less than 2, it must lie between -2 and 2, so $-2 < x-3 < 2$. Now $-2+3 < x < 2+3$, $1 < x < 5$.

METHOD 3: Since both sides of the inequality are positive or zero, we may square both sides, and the sign of the inequality will be the same. So

$$|x-3|^2 < 2^2, \quad (x-3)^2 < 4, \quad x^2-6x+9 < 4, \quad x^2-6x+5 < 0, \quad (x-1)(x-5) < 0.$$

Here we found the roots of $x^2-6x+5=0$ using the quadratic formula: $x_1 = \frac{6+\sqrt{6^2-4\cdot 5}}{2} = \frac{6+\sqrt{36-20}}{2} = \frac{6+\sqrt{16}}{2} = \frac{6+4}{2} = 5$ and $x_2 = \frac{6-\sqrt{16}}{2} = \frac{6-4}{2} = 1$. Now test the sign:

$$\underline{\quad + \quad}_{1}\underline{\quad - \quad}_{5}\underline{\quad + \quad}$$

Answer: $x \in (1,5)$.

(b)
$$x-1 \leq \frac{4}{x-1}$$

Solution: First bring everything to one side:

$$x-1-\frac{4}{x-1} \leq 0.$$

Simplify:

$$\frac{(x-1)^2-4}{x-1} \leq 0; \quad \frac{x^2-2x+1-4}{x-1} \leq 0; \quad \frac{x^2-2x-3}{x-1} \leq 0; \quad \frac{(x+1)(x-3)}{x-1} \leq 0.$$

Testing:

$$\underline{\quad - \quad}_{-1}\underline{\quad + \quad}_{\hat{1}}\underline{\quad - \quad}_{3}\underline{\quad + \quad}$$

The point $x=1$ does not belong to the solution set, since it is not in the domain.

Answer: $x \in (-\infty, -1] \cup (1,3]$.

6. Find the following limits: [15]

(a)
$$\lim_{x\to-\infty} \frac{3x+\sin x}{2x^2+x+1}$$

Solution: The limit type is "$\frac{-\infty}{\infty}$", thus it is indefinite type. So we must divide the numerator and denominator by x^2 which is the highest power of x:

$$= \lim_{x\to-\infty} \frac{\frac{3}{x}+\frac{\sin x}{x^2}}{2+\frac{1}{x}+\frac{1}{x^2}}$$

Let us find the limit of $\frac{\sin x}{x^2}$ as $x \to -\infty$. Since $-\frac{1}{x^2} \le \frac{\sin x}{x^2} \le \frac{1}{x^2}$ for $x \neq 0$, and the "squeezing" functions $-\frac{1}{x^2}$ and $\frac{1}{x^2}$ have the same limit 0 as $x \to -\infty$, the function $\frac{\sin x}{x^2}$ is forced to have the same limit by the **Squeeze Theorem**. So, $\lim_{x\to-\infty} \frac{\sin x}{x^2} = 0$.

Thus
$$= \frac{\text{“}\frac{3}{-\infty}\text{”}+0}{2+\text{“}\frac{1}{-\infty}\text{”}+\text{“}\frac{1}{\infty}\text{”}} = \frac{0+0}{2+0+0} = 0.$$

Answer: 0.

(b)
$$\lim_{x\to2} \frac{x^3-x^2-4}{x^2-4}$$

Solution: First establish the type of limit:

$$= \text{“}\frac{8-4-4}{4-4}\text{”} = \text{“}\frac{0}{0}\text{”}.$$

Since the limit has indefinite type, we must simplify the function. Since $x=2$ is a root of the numerator, $x-2$ is a factor of the numerator. To find another factor, we use long division of polynomials, thus $x^3-x^2-4 = (x-2)(x^2+x+2)$. So

$$\lim_{x\to2} \frac{x^3-x^2-4}{x^2-4} = \lim_{x\to2} \frac{(x-2)(x^2+x+2)}{(x-2)(x+2)} = \lim_{x\to2} \frac{x^2+x+2}{x+2}$$

Now we may substitute $x=2$:

$$= \frac{4+2+2}{2+2} = 2.$$

Answer: 2.

7. Given the data in the following table, determine $(f \circ g)'(2)$ [9]

x	-1	0	2	5
$f(x)$	7	1	5	7
$g(x)$	5	3	0	4
$f'(x)$	0	2	3	-1
$g'(x)$	4	0	-1	6

Solution: By the Chain Rule, $(f \circ g)'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x)$.
Substitute $x = 2$:

$$(f \circ g)'(2) = f'(g(2)) \cdot g'(2)$$

From the table, we get $g(2) = 0$ and $g'(2) = -1$, thus

$$= f'(0) \cdot (-1)$$

From the table, we get $f'(0) = 2$, thus

$$= 2 \cdot (-1) = -2.$$

Answer: -2.

8. Find the equation of the tangent line to the graph of [15]

$$\cos(xy) + \sin(y^2) + x^3 - xy = 9$$

at the point $(2, 0)$.

Solution: First check that the point $(2, 0)$ satisfies the given equation: for $x = 2, y = 0$ the left hand side equals $\cos(0) + \sin(0) + 8 - 0 = 1 + 0 + 8 - 0 = 9$, thus the point $(2, 0)$ belongs to the graph.

Let us differentiate both sides of $\cos(xy) + \sin(y^2) + x^3 - xy = 9$ with respect to x using the chain rule, remembering that y is a function of x:

$$-\sin(xy) \cdot (y + xy') + \cos(y^2) \cdot 2yy' + 3x^2 - (y + xy') = 0.$$

In order to solve this equation for y', we bring all terms containing y' to the left, and bring all terms which do not contain y' to the right:

$$(-x\sin(xy) + 2y\cos(y^2) - x)y' = y\sin(xy) - 3x^2 + y, \quad y' = \frac{y\sin(xy) - 3x^2 + y}{-x\sin(xy) + 2y\cos(y^2) - x}.$$

Substituting $x = 2, y = 0$, we obtain the slope of the tangent line at the point $(2, 0)$:

$$m = y'(2) = \left. \frac{y\sin(xy) - 3x^2 + y}{-x\sin(xy) + 2y\cos(y^2) - x} \right|_{\substack{x=2 \\ y=0}} = \frac{0 - 3 \cdot 4 + 0}{0 + 0 - 2} = \frac{-12}{-2} = 6.$$

So, the tangent line at the point $(2, 0)$ has slope $m = 6$. Therefore the equation of this tangent line is $y = m(x - 2) + 0 = 6(x - 2) + 0 = 6x - 12$.

Answer: $y = 6x - 12$.