MATH 251 B47/B48/B50 Quiz #2 Winter 2005

Name: K E Y I.D.#: ____________

Answer all questions. Calculators are NOT allowed. 30 minutes.

1. Answer each of the following questions. No explanation is necessary. [35]

(a) Let $f(x) = \begin{cases} |x-2| & \text{if } x \leq 1, \\ k - |x| & \text{if } x > 1. \end{cases}$
Find k so that f is continuous at $x = 1$. **2**

(b) Answer TRUE or FALSE: If f, g are discontinuous at $x = a$ then $f - g$ is discontinuous at $x = a$. **FALSE**

(c) For which value of x does the function $f(x) = \frac{x^2-3x+2}{x^2-4}$ have a removable discontinuity? **2**

(d) Find any vertical asymptote for $f(x) = \frac{x^2-3x+2}{x^2-4}$. **$x = -2$**

(e) How many horizontal asymptotes has $g(x) = \frac{x^4+\sqrt{x^2+1}}{x^4-5x^2+4}$? **1**

(f) The Intermediate-Value Theorem allows us to conclude that the equation $x^3 + 2x - 10 = 0$ has a root in the interval
a: $[-1, 0]$, b: $[0, 1]$, c: $[1, 2]$, d: None of these. **c**

(g) Let r denote a root of the equation $x^3 + 2x - 10 = 0$.
Find an estimate for r with Error $< \frac{1}{4}$.
Hint: Use the bisection method. **1.75**

2. Determine $\displaystyle\lim_{x\to-\infty} \frac{1-3x}{2x-\sin^2 x}$. Show all work. [15]

Solution: The type of limit is "$\frac{\infty}{-\infty}$". Divide both the numerator and denominator by the highest power of x:

$$\lim_{x\to-\infty} \frac{1-3x}{2x-\sin^2 x} = \lim_{x\to-\infty} \frac{\frac{1}{x}-3}{2-\frac{\sin^2 x}{x}}.$$

Let us find the limit of $\frac{\sin^2 x}{x}$ as $x \to -\infty$. Since $0 \geq \frac{\sin^2 x}{x} \geq \frac{1}{x}$ for $x < 0$, and the "squeezing" functions $\frac{1}{x}$ and 0 have the same limit 0 as $x \to -\infty$, the function $\frac{\sin^2 x}{x}$ is forced to have the same limit by the Squeeze Theorem. So, $\lim_{x\to-\infty} \frac{\sin^2 x}{x} = 0$ and

$$\lim_{x\to-\infty} \frac{\frac{1}{x}-3}{2-\frac{\sin^2 x}{x}} = \frac{0-3}{2-0} = -\frac{3}{2}.$$

TURN OVER FOR QUESTIONS 3 AND 4

3. For $f(x) = \frac{x^2-x}{x^3+2x^2-x-2}$ find [30]

$$\text{a: } \lim_{x\to -2^-} f(x), \qquad \text{b: } \lim_{x\to -\infty} f(x), \qquad \text{c: } \lim_{x\to 1} f(x).$$

Solution: First establish the types of limits:

$$\text{a: } \text{“}\frac{4+2}{-8+8+2-2}\text{”} = \text{“}\frac{6}{0}\text{”}, \qquad \text{b: } \text{“}\frac{\infty}{-\infty}\text{”}, \qquad \text{c: } \text{“}\frac{1-1}{1+2-1-2}\text{”} = \text{“}\frac{0}{0}\text{”}.$$

We see that the limit in (a) DOES NOT EXIST, but we still must find out whether it is ∞ or $-\infty$ or neither. Since all limits have indefinite types, we must simplify $f(x)$. While calculating the types of limits, we found out that the denominator has roots at $x = -2$ and $x = 1$. Therefore $(x+2)$ and $(x-1)$ are factors of the denominator, so $x^3+2x^2-x-2 = (x+2)(x-1)(x+1)$. Cancel the common factor: $f(x) = \frac{x(x-1)}{(x+2)(x-1)(x+1)} = \frac{x}{(x+2)(x+1)}$ for $x \neq 1$. Now

for a: $$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{x}{(x+2)(x+1)} = \text{“}\frac{-2}{(0^-)\cdot(-1)}\text{”} = -\infty, \qquad \textbf{DNE}.$$

for b: $$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x}{(x+2)(x+1)} = \frac{\frac{1}{x}}{(1+\frac{2}{x})(1+\frac{1}{x})} = \frac{0}{(1+0)(1+0)} = 0.$$

for c: $$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x}{(x+2)(x+1)} = \frac{1}{(1+2)(1+1)} = \frac{1}{3\cdot 2} = \frac{1}{6}.$$

4. Find the limit and state your answer as ∞ or $-\infty$ if appropriate. [20]

$$\lim_{x\to -\infty} \frac{1}{\sqrt{x^2+x+1} - \sqrt{x^2-3x}}.$$

Solution: The limit has type “$\frac{1}{\infty-\infty}$”. Since it is indefinite type, we must simplify the function. Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{1}{\sqrt{x^2+x+1} - \sqrt{x^2-3x}} = \frac{\sqrt{x^2+x+1} + \sqrt{x^2-3x}}{(\sqrt{x^2+x+1} - \sqrt{x^2-3x})(\sqrt{x^2+x+1} + \sqrt{x^2-3x})}$$

$$= \frac{\sqrt{x^2+x+1} + \sqrt{x^2-3x}}{x^2+x+1-(x^2-3x)} = \frac{\sqrt{x^2+x+1} + \sqrt{x^2-3x}}{4x+1} = \text{“}\frac{\infty}{-\infty}\text{”}$$

and the limit is indefinite. Therefore we must divide the denominator and numerator by the highest power of x in the denominator:

$$= \frac{\frac{\sqrt{x^2+x+1}}{x} + \frac{\sqrt{x^2-3x}}{x}}{4+\frac{1}{x}}$$

Since $x \to -\infty$, x is negative, so we have $x = -\sqrt{x^2}$ (with the minus sign), so

$$= \frac{\frac{\sqrt{x^2+x+1}}{-\sqrt{x^2}} + \frac{\sqrt{x^2-3x}}{-\sqrt{x^2}}}{4+\frac{1}{x}} = \frac{-\sqrt{\frac{x^2+x+1}{x^2}} - \sqrt{\frac{x^2-3x}{x^2}}}{4+\frac{1}{x}} = \frac{-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - \sqrt{1-\frac{3}{x}}}{4+\frac{1}{x}}.$$

Now

$$\lim_{x\to -\infty} (\sqrt{x^2+x+1} - \sqrt{x^2-3x}) = \lim_{x\to -\infty} \frac{-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - \sqrt{1-\frac{3}{x}}}{4+\frac{1}{x}} = \lim_{x\to -\infty} \frac{-1-1}{4} = -\frac{1}{2}.$$

END OF PAPER