MATH 251 B45/B46 Quiz #2 Winter 2005

Name: K E Y I.D.#: ____________

Answer all questions. Calculators are NOT allowed. 30 minutes.

1. Answer each of the following questions. No explanation is necessary. [35]

   (a) Let $f(x) = \begin{cases} \lfloor x^2 \rfloor & \text{if } x < 2, \\ k & \text{if } x \geq 2. \end{cases}$
   Find k so that f is continuous at $x = 2$. ___3___

   (b) Answer TRUE or FALSE: If f, g are discontinuous at $x = a$ then $f + g$ is discontinuous at $x = a$. ___FALSE___

   (c) For which value of x does the function $f(x) = \frac{x^2+3x+2}{x^2-1}$ have a removable discontinuity? ___-1___

   (d) Find any vertical asymptote for $f(x) = \frac{x^2+3x+2}{x^2-1}$. ___$x = 1$___

   (e) How many horizontal asymptotes has $g(x) = \frac{4x^2+3x-8}{x^2+\sqrt{x^2+1}}$? ___1___

   (f) The Intermediate-Value Theorem allows us to conclude that the equation $x^3 + x - 3 = 0$ has a root in the interval
   a: $[-1, 0]$, b: $[0, 1]$, c: $[1, 2]$, d: None of these. ___c___

   (g) Let r denote a root of the equation $x^3 + x - 3 = 0$.
   Find an estimate for r with Error $< \frac{1}{4}$.
   Hint: Use the bisection method. ___1.25___

2. Determine $\displaystyle\lim_{x\to-\infty} \frac{x + \sin^2 x}{2 - 3x}$. Show all work. [15]

   Solution: The type of limit is "$\frac{-\infty}{\infty}$". Divide both the numerator and denominator by the highest power of x:

$$\lim_{x\to-\infty} \frac{x + \sin^2 x}{2 - 3x} = \lim_{x\to-\infty} \frac{1 + \frac{\sin^2 x}{x}}{\frac{2}{x} - 3}.$$

   Let us find the limit of $\frac{\sin^2 x}{x}$ as $x \to -\infty$. Since $0 \geq \frac{\sin^2 x}{x} \geq \frac{1}{x}$ for $x < 0$, and the "squeezing" functions $\frac{1}{x}$ and 0 have the same limit 0 as $x \to -\infty$, the function $\frac{\sin^2 x}{x}$ is forced to have the same limit by the Squeeze Theorem. So, $\lim_{x\to-\infty} \frac{\sin^2 x}{x} = 0$ and

$$\lim_{x\to-\infty} \frac{1 + \frac{\sin^2 x}{x}}{\frac{2}{x} - 3} = \frac{1+0}{0-3} = -\frac{1}{3}.$$

TURN OVER FOR QUESTIONS 3 AND 4

3. For $f(x)=\frac{x^2-4}{x^3-7x-6}$ find [30]

$$\text{a: } \lim_{x\to -1^-} f(x), \qquad \text{b: } \lim_{x\to -\infty} f(x), \qquad \text{c: } \lim_{x\to -2} f(x).$$

Solution: First establish the types of limits:

$$\text{a: } \text{“}\frac{1-4}{-1+7-6}\text{”} = \text{“}\frac{-3}{0}\text{”}, \qquad \text{b: } \text{“}\frac{\infty}{-\infty}\text{”}, \qquad \text{c: } \text{“}\frac{4-4}{-8+14-6}\text{”} = \text{“}\frac{0}{0}\text{”}.$$

We see that the limit in (a) DOES NOT EXIST, but we still must find out whether it is ∞ or $-\infty$ or neither. Since all limits have indefinite types, we must simplify $f(x)$. While calculating the types of limits, we found out that the denominator has roots at $x=-1$ and $x=-2$. Therefore $(x+1)$ and $(x+2)$ are factors of the denominator, so $x^3-7x-6=(x+1)(x+2)(x-3)$. Cancel the common factor: $f(x)=\frac{(x-2)(x+2)}{(x+1)(x+2)(x-3)}=\frac{x-2}{(x+1)(x-3)}$ for $x\neq -2$. Now

for a:
$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\frac{x-2}{(x+1)(x-3)} = \text{“}\frac{-3}{(0^-)\cdot(-4)}\text{”} = -\infty, \qquad \textbf{DNE}.$$

for b:
$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty}\frac{x-2}{(x+1)(x-3)} = \frac{\frac{1}{x}-\frac{2}{x^2}}{(1+\frac{1}{x})(1-\frac{3}{x})} = \frac{0-0}{(1+0)(1-0)} = 0.$$

for c:
$$\lim_{x\to -2} f(x) = \lim_{x\to -2}\frac{x-2}{(x+1)(x-3)} = \frac{-2-2}{(-2+1)(-2-3)} = \frac{-4}{(-1)(-5)} = -\frac{4}{5}.$$

4. Find the limit and state your answer as ∞ or $-\infty$ if appropriate. [20]

$$\lim_{x\to -\infty}(\sqrt{x^2+2x}-\sqrt{x^2-2x}).$$

Solution: The limit has type “$\infty-\infty$”. Since it is indefinite type, we must simplify the function. Multiply the numerator and denominator by the conjugate of the numerator:

$$\sqrt{x^2+2x}-\sqrt{x^2-2x} = \frac{(\sqrt{x^2+2x}-\sqrt{x^2-2x})(\sqrt{x^2+2x}+\sqrt{x^2-2x})}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

$$= \frac{x^2+2x-(x^2-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}} = \frac{4x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}} = \text{“}\frac{-\infty}{\infty}\text{”}$$

and the limit is indefinite. Therefore we must divide the denominator and numerator by the highest power of x in the denominator:

$$= \frac{4}{\frac{\sqrt{x^2+2x}}{x}+\frac{\sqrt{x^2-2x}}{x}}$$

Since $x\to -\infty$, x is negative, so we have $x=-\sqrt{x^2}$ (with the minus sign), so

$$= \frac{4}{\frac{\sqrt{x^2+2x}}{-\sqrt{x^2}}+\frac{\sqrt{x^2-2x}}{-\sqrt{x^2}}} = \frac{4}{-\sqrt{\frac{x^2+2x}{x^2}}-\sqrt{\frac{x^2-2x}{x^2}}} = \frac{4}{-\sqrt{1+\frac{2}{x}}-\sqrt{1-\frac{2}{x}}}.$$

Now

$$\lim_{x\to -\infty}(\sqrt{x^2+2x}-\sqrt{x^2-2x}) = \lim_{x\to -\infty}\frac{4}{-\sqrt{1+\frac{2}{x}}-\sqrt{1-\frac{2}{x}}} = \lim_{x\to -\infty}\frac{4}{-1-1} = -2.$$

END OF PAPER