MATH 251 B47/B48/B50 Quiz #3 Winter 2005

Name: K E Y I.D.#:

Answer all questions. Calculators are NOT allowed. 30 minutes.

1. Find the equation of the tangent line to the graph of [25]
   $y\cos(x+y) = x + y - \frac{\pi}{2}$ at the point $(0, \frac{\pi}{2})$.

   Solution: First check that the point $(0, \frac{\pi}{2})$ satisfies the given relation: $\cos\frac{\pi}{2} = 0 + \frac{\pi}{2} - \frac{\pi}{2}$, so the point $(0, \frac{\pi}{2})$ belongs to the graph. Let us differentiate both sides of $y\cos(x+y) = x + y - \frac{\pi}{2}$ with respect to x using the chain rule, remembering that y is a function of x:

   $$y'\cos(x+y) + y(-\sin(x+y))(1+y') = 1 + y' - 0,$$

   $$y'(\cos(x+y) - y\sin(x+y) - 1) = 1 + y\sin(x+y),$$

   $$y' = \frac{1 + y\sin(x+y)}{\cos(x+y) - y\sin(x+y) - 1},$$

   Substituting $x = 0, y = \frac{\pi}{2}$, we obtain the slope of the tangent line at the point $(0, \frac{\pi}{2})$:

   $$m = y'(0) = \left.\frac{1 + \frac{\pi}{2}\sin(0 + \frac{\pi}{2})}{\cos(0+\frac{\pi}{2}) - \frac{\pi}{2}\sin(0+\frac{\pi}{2}) - 1}\right|_{\substack{x=0\\ y=\frac{\pi}{2}}} = \frac{1 + \frac{\pi}{2}}{0 - \frac{\pi}{2} - 1} = -1.$$

   So, the tangent line at the point $(0, \frac{\pi}{2})$ has slope $m = -1$. Therefore the equation of this tangent line is $y = mx + \frac{\pi}{2} = -x + \frac{\pi}{2}$.

   Answer: $y = -x + \frac{\pi}{2}$.

2. Find the indefinite integrals: [25]

   (a) $\int \frac{x^2+4}{2x}\,dx = \int \frac{x}{2} + \frac{2}{x}\,dx = \frac{x^2}{4} + 2\ln|x| + C$

   (b) $\int 2x\sin(x^2)\,dx = \int (x^2)'\sin(x^2)\,dx = -\cos(x^2) + C$

TURN OVER FOR QUESTIONS 3, 4, AND 5

3. Let g be the inverse function of f, that is $g(f(x)) = x$ for all $x \in D_f$ and $f(g(y)) = y$ for all $y \in D_g$. Calculate g if [20]

$$f(x) = \frac{2x-1}{x+2} .$$

Specify the domains and ranges of f and g.

Solution: The function f is defined for any $x \neq -2$, so its domain is $D_f = (-\infty, -2) \cup (-2, \infty)$. Since $f(x) = \frac{2x-1}{x+2} = \frac{2x+4-5}{x+2} = 2 - \frac{5}{x+2}$, the graph of f is a hyperbola with the vertical asymptote $x = -2$ and the horizontal asymptote $y = 2$. So, any number $y \neq 2$ is an output value of $y = f(x)$. Thus the range of f is $R_f = (-\infty, 2) \cup (2, \infty)$. Now

$$R_g = D_f = (-\infty, -2) \cup (-2, \infty), \qquad D_g = R_f = (-\infty, 2) \cup (2, \infty).$$

Denote $y = f(x) = \frac{2x-1}{x+2}$. To find the inverse function g, we must rewrite this equation in the form $x = g(y)$. So, we must solve the equation $y = \frac{2x-1}{x+2}$ for x:

$$y = \frac{2x-1}{x+2}, \qquad y(x+2) = 2x - 1, \qquad (y-2)x = -2y - 1, \qquad x = \frac{-2y-1}{y-2}.$$

Therefore, $g(y) = \frac{-2y-1}{y-2}$.

4. Differentiate $\quad \ln|\sec x + \tan x|$ [15]

Solution:

$$(\ln|\sec x + \tan x|)' = \frac{1}{\sec x + \tan x} \cdot (\sec x + \tan x)'$$

$$= \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x) = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

5. Simplify [15]

$e^{2\ln 7} + 2\ln\left(\frac{1}{e^7}\right) = e^{(\ln 7)\cdot 2} + 2\ln(e^{-7}) = (e^{\ln 7})^2 + 2\ln(e^{-7})$

OR

$e^{2\ln 7} + 2\ln\left(\frac{1}{e^7}\right) = e^{\ln(7^2)} + 2(\ln 1 - \ln(e^7)) = e^{\ln 49} + 2(0 - \ln(e^7)) = e^{\ln 49} - 2\ln(e^7)$

Since the functions $\ln x$ and e^x are the inverses of each other, $\ln(e^x) = x$ and $e^{\ln x} = x$. So,

$$= 7^2 + 2(-7) = 49 - 14 = 35$$

OR

$$= 49 - 2 \cdot 7 = 49 - 14 = 35.$$

Answer: 35.

END OF PAPER