MATH 251 B45/B46 Quiz #3 Winter 2005

Name: K E Y I.D.#: ____________

Answer all questions. Calculators are NOT allowed. 30 minutes.

1. Find the equation of the tangent line to the graph of [25]
$xy^2 - x^2y = 16$ at the point $(2, -2)$.

Solution: First check that the point $(2, -2)$ satisfies the given relation: $2 \cdot 4 - 4(-2) = 8 + 8 = 16$, so the point $(2, -2)$ belongs to the graph. Let us differentiate both sides of $xy^2 - x^2y = 16$ with respect to x using the chain rule, remembering that y is a function of x:

$$y^2 + x \cdot 2yy' - 2xy - x^2y' = 0, \qquad y'(2xy - x^2) = 2xy - y^2, \qquad y' = \frac{2xy - y^2}{2xy - x^2},$$

Substituting $x = 2, y = -2$, we obtain the slope of the tangent line at the point $(2, -2)$:

$$m = y'(2) = \frac{2xy - y^2}{2xy - x^2}\Bigg|_{\substack{x=2 \\ y=-2}} = \frac{-8-4}{-8-4} = 1.$$

So, the tangent line at the point $(2, -2)$ has slope $m = 1$. Therefore the equation of this tangent line is $y = m(x - 2) - 2 = x - 4$.

Answer: $y = x - 4$.

2. Find the indefinite integrals: [25]

(a) $\int x^{10}\, dx = \frac{x^{11}}{11} + C$

(c) $\int (x^{1/3} + \sec^2 x + e^x)\, dx = \frac{3}{4}x^{4/3} + \tan x + e^x + C$

TURN OVER FOR QUESTIONS 3, 4, AND 5

3. Let g be the inverse function of f. Calculate g if [20]

$$f(x) = \frac{x+1}{x-1} \ .$$

Specify the domains and ranges of f and g.

Solution: The function f is defined for any $x \neq 1$, so its domain is $D_f = (-\infty, 1) \cup (1, \infty)$. Since $f(x) = \frac{x+1}{x-1} = \frac{x-1+2}{x-1} = 1 + \frac{2}{x-1}$, the graph of f is a hyperbola with the vertical asymptote $x = 1$ and the horizontal asymptote $y = 1$. So, any number $y \neq 1$ is an output value of $y = f(x)$. Thus the range of f is $R_f = (-\infty, 1) \cup (1, \infty)$. Now

$$R_g = D_f = (-\infty, 1) \cup (1, \infty), \qquad D_g = R_f = (-\infty, 1) \cup (1, \infty).$$

Denote $y = f(x) = \frac{x+1}{x-1}$. To find the inverse function g, we must rewrite this equation in the form $x = g(y)$. So, we must solve the equation $y = \frac{x+1}{x-1}$ for x:

$$y = \frac{x+1}{x-1}, \qquad y(x-1) = x+1, \qquad (y-1)x = y+1, \qquad x = \frac{y+1}{y-1}.$$

Therefore, $g(y) = \frac{y+1}{y-1}$.

4. Differentiate $\quad x^2 \ln(e^x + 1)$ [15]

Solution:

$$\begin{aligned}(x^2 \ln(e^x+1))' &= (x^2)' \ln(e^x+1) + x^2 \left(\ln(e^x+1)\right)' \\ &= 2x \ln(e^x+1) + x^2 \cdot \frac{e^x}{e^x+1}\end{aligned}$$

5. Simplify [15]

$5e^{-\ln 5} - \ln(e^{-5}) = 5e^{(\ln 5)(-1)} - \ln(e^{-5}) = 5(e^{\ln 5})^{-1} - \ln(e^{-5})$

OR

$5e^{-\ln 5} - \ln(e^{-5}) = 5e^{\ln(5^{-1})} - \ln(e^{-5})$

Since the functions $\ln x$ and e^x are the inverses of each other, $\ln(e^x) = x$ and $e^{\ln x} = x$. So,

$$= 5 \cdot 5^{-1} - (-5) = \frac{5}{5} + 5 = 1 + 5 = 6.$$

Answer: 6.

END OF PAPER