MATH 251 B47/B48/B50 Quiz #4 Winter 2005

Name: ___K E Y___ **I.D.#:** ________

Answer all questions. Calculators are NOT allowed. 30 minutes.

1. Simplify: [10]

   (a) $$\log_3 81 = \log_3(3^4) = 4.$$

   (b) $$(\log_{10} 2)\ (\log_2 0.01) = \log_{10} 0.01 = \log_{10}(10^{-2}) = -2$$

2. Differentiate: [20]

   (a) $\quad y = x^3 \arctan x, \qquad y' = 3x^2 \arctan x + x^3 \dfrac{1}{1+x^2} = 3x^2 \arctan x + \dfrac{x^3}{1+x^2}.$

   (b) $$y = x^{\ln x}, \qquad \ln y = (\ln x)(\ln x) = (\ln x)^2,$$
   $$\frac{y'}{y} = 2(\ln x)\frac{1}{x} = 2\frac{\ln x}{x},$$
   $$y' = 2y\frac{\ln x}{x} = 2x^{\ln x}\frac{\ln x}{x} = 2x^{\ln x - 1}\ln x.$$

3. A rectangle has two vertices on the x-axis and the other two on the semicircle $y = \sqrt{2 - x^2}$. Determine its **maximal area**. [20]

Solution: The rectangle has width $2x$ and height $y = \sqrt{2 - x^2}$, so the area equals
$$A(x) = 2xy = 2x\sqrt{2 - x^2}.$$

We want to minimize the function $A(x)$ on $[0, \sqrt{2}]$. Since the area is always posiitve or 0, minimizing the function $A(x)$ on $[0, \sqrt{2}]$ is the same as minimizing the function
$$B(x) = (A(x))^2 = 4x^2(2 - x^2) = 4(2x^2 - x^4)$$

on $[0, \sqrt{2}]$. Since the function B is a polynomial, it is differentiable, so it has **no singular points**. In order to find critical points of B, we must solve the equation $B'(x) = 0$ for x:
$$B'(x) = 4(4x - 4x^3) = 16(x - x^3) = 16x(1 - x)(1 + x),$$

so the only **critical point** in $(0, \sqrt{2})$ is $x = 1$ (since $0 < 1 < \sqrt{2}$, and the points $x = 0$ and $x = -1$ do not belong to the open interval $(0, \sqrt{2})$). Let us find the **type** of this critical point.

The **1st derivative test:**

x	$(0,1)$	$(1,\sqrt{2})$
$B'(x)$	$+$	$-$
B	↗	↘

$\Longrightarrow$ **local max at** $x = 1$

OR The **2nd derivative test:** $B'(1) = 0,\ B''(x) = 16(1-3x^2),\ B''(1) = 16(1-3) < 0$, so the function B is **concave down**, so **local maximum**.

Thus, B has an **absolute maximum** at the critical point $x = 1$.
The maximal area equals $\quad A_{max} = \sqrt{B_{max}} = A(1) = 2\sqrt{2 - 1} = 2.$

Answer: $A_{max} = 2.$

4. Sketch the graph of $y = f(x) = xe^x$. [30]

(a) Find the domain, range, asymptotes.

(b) Find the intervals where f is increasing or decreasing.

(c) Find the intervals where f is concave up or concave down.

(d) Find the (x, y)-coordinates of all local minima, local maxima, inflexion points, x- and y-intercepts, and indicate them on the graph.

Solution: (a) **Domain:** $D_f = (-\infty, \infty) = \mathbb{R}$, since e^x is defined for any x.

Vertical asymptotes: none, since f is continuous everywhere.

Horizontal asymptote:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} xe^x = \text{“}\infty \cdot \infty\text{”} = \text{“}\infty\text{”}, \qquad \text{DNE},$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} xe^x = \text{“}(-\infty)(0^+)\text{”} = \text{“}(-1)(0^+)\text{”} = 0^-,$$

since “in a battle between a polynomial and an exponent, the exponent always wins”; so $y = 0$ is a **horizontal asymptote such that** $y \to 0^-$ **as** $x \to -\infty$.

Slant asymptotes: There still can be a slant (diagonal) asymptote $y = ax + b$ for $x \to \infty$:

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^x = \infty, \qquad \text{DNE},$$

so the function f increases much faster than any straight line as $x \to \infty$, so **no slant asymptotes**.

Now draw a small part of the graph near the asymptote!

(b)
$$f'(x) = e^x + xe^x = (1 + x)e^x,$$

the **1st derivative test**:

x	$(-\infty, -1)$	$(-1, \infty)$
$f'(x)$	$-$	$+$
f	$\searrow$	$\nearrow$

$\Longrightarrow$ **local min at $x = -1$**

(c)
$$f''(x) = e^x + (1 + x)e^x = (2 + x)e^x,$$

the **2nd derivative test**:

x	$(-\infty, -2)$	$(-2, \infty)$
$f''(x)$	$-$	$+$
f	$\frown$	$\smile$

$\Longrightarrow$ **inflexion point at $x = -2$**

(d) **local maxima:** none, **local minimum:** $(-1, -\frac{1}{e})$ (by part (b));
inflexion point: $(-2, -\frac{2}{e^2})$ (by part (c));
x-intercept: $(0, 0)$ (by solving equation $f(x) = 0$ for x);
y-intercepts: $(0, 0)$ (by taking $f(0)$).

Sketching the graph: Indicate the intercepts and the points of local maxima and minima, and join them, by monotone curves, with the parts of the graph (drawn in part (a)) near the asymptotes. **Make sure that no additional intercepts appear, and the intervals where f is increasing or decreasing are exactly those found in part (b)!** Indicate the inflexion points on the graph.

(a) **Range:** By part (b), f is decreasing on $(-\infty, -1]$ from 0^- up to $-\frac{1}{e}$, and increasing on $[-1, \infty)$ from $-\frac{1}{e}$ up to ∞ (see also part (a)). So, the range is $R_f = [-\frac{1}{e}, \infty)$. (This can also be seen from the graph.)

5. (a) Approximate $\sqrt{17}$ using linearization. [20]

(b) Determine the sign of the Error, and estimate its size.
Is this estimate an overestimate or underestimate?

Solution: Since $\sqrt{16} = 4$, it is useful to approximate the function $y = f(x) = \sqrt{x}$ near the point $a = 16$. Let us approximate the graph of $y = f(x)$ by its tangent line at the point $(a, f(a)) = (16, 4)$. We need two derivatives of f:

$$f(x) = \sqrt{x}, \qquad f'(x) = \frac{1}{2\sqrt{x}}, \qquad f''(x) = -\frac{1}{4\sqrt{x^3}}.$$

The tangent line at the point $(16, 4)$ has slope $m = f'(16) = \frac{1}{8}$. The equation of the line having slope $m = \frac{1}{8}$ and going through the point $(16, 4)$ is

$$y = 4 + m \cdot (x - 16), \qquad y = 4 + \frac{x - 16}{8}.$$

The function $L(x) = 4 + \frac{x-16}{8}$ on the right hand side of this equation is called the **linearization of f at the point a**. We have

$$\sqrt{17} = f(17) \approx L(17) = 4 + \frac{1}{8} = 4.125$$

(b) The Error $E(x) = f(x) - L(x)$ of the linear approximation has the following form:

$$E(x) = \frac{f''(c)}{2}(x - a)^2 \qquad \text{for some } c \in (a, x).$$

(Here we used that $a = 16 < 17 = x$.) Therefore

$$E(17) = \frac{f''(c)}{2} 1^2 = -\frac{1}{8c^{3/2}} \quad \text{for some } c \in (16, 17),$$

so $E(17) < 0$. Since (ERROR)=(TRUE VALUE)−(APPROXIMATION)< 0, our approximation 4.125 is an **overestimate** for $\sqrt{17}$.

Let us estimate the **absolute value of the Error**, using the inequality $16 < c < 17$:

$$|E(17)| = \frac{1}{8c^{3/2}} < \frac{1}{8 \cdot 16^{3/2}} = \frac{1}{8 \cdot 4^3} = \frac{1}{512} < \frac{1}{500} = 0.002$$

Answer: $\sqrt{17} \approx 4.125$, $\quad E(17) < 0$, $\quad |E(17)| < .002$, $\quad$ an overestimate.